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OBSERVATION ON THE NEGATIVE PELLIAN EQUATION

$$y^2 = 180x^2 - 11$$

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ABSTRACT:

The binary quadratic equation represented by the negative pellian $y^2 = 180x^2 - 11$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle. Further, employing the integer solutions of the negative pellian equation under consideration, a few interesting relations between special polygonal numbers are obtained.

KEYWORDS: Binary quadratic; hyperbola; parabola; integral solutions; pell equation; polygonal numbers.

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NOTATIONS:

$P_n^m = \frac{[n(n+1)(m-2)n+(5-m)]}{6}$ = Pyramidal number of rank n with size m

$pr_n = n(n+1)$ = Pronic number of rank n

$t_{m,n} = n\frac{(n+1)}{2}$ = Triangular number of rank n size m

INTRODUCTION: Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity , J.L. Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solution whereas $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2- 14].More specifically, one may refer “ The On-line Encyclopedia of integer sequences ” (A031396,A130226,A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative Pell equation given by $y^2 = 180x^2 - 11$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

METHODS OF ANALYSIS:

Consider the binary quadratic equation

$$y^2 = 180x^2 - 11 \quad (1)$$

with the least positive integer solution $x_0 = 1$, $y_0 = 13$

To obtain the other solutions of (1), Consider the pellian equation

$$y^2 = 180x^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{g_n}{12\sqrt{5}}, \quad \tilde{y}_n = \frac{f_n}{2}$$

in which, $f_n = \left(161 + 72\sqrt{5}\right)^{n+1} + \left(161 - 72\sqrt{5}\right)^{n+1}$

$$g_n = \left(161 + 72\sqrt{5}\right)^{n+1} - \left(161 - 72\sqrt{5}\right)^{n+1}$$

Applying Brahmagupta lemma between the solutions of

(x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the general solution of (1) is found to be

$$x_{n+1} = \frac{f_n}{2} + \frac{13}{60}\sqrt{5}g_n \qquad (2)$$

$$y_{n+1} = \frac{13f_n}{2} + 3\sqrt{5}g_n \qquad (3)$$

where n=-1, 0, 1, 2. ……….

Thus, (2) and (3) represent non-zero distinct integral solutions of (1) which represent a hyperbola.

The recurrence relations satisfied by the values of x and y are respectively

$$322x_{n+2} - x_{n+1} - x_{n+3} = 0$$

$$322y_{n+2} - y_{n+1} - y_{n+3} = 0$$

A few numerical examples are presented in the table below:

n	x_{n+1}	y_{n+1}
-1	1	13
0	317	4253
1	102073	1369453
2	32867189	440959613
3	10583132785	141987625933
4	3407735889581	45719574590813

A few interesting relations among the solutions are presented below:

- Both the values of x and y are odd
- $4253x_{3n+3}-13x_{3n+4}\cong 0(\text{mod } 66)$
- $1369453x_{n+1}-13x_{n+3}\cong 0(\text{mod } 21252)$
- $(y_{n+2}-317y_{n+1})\cong 0(\text{mod } 11)$
- $4253x_{n+1}-13x_{n+2}=1369453x_{n+2}-4253x_{n+3}$
- $x_{n+2}y_{n+1}-x_{n+1}y_{n+2}=-132$
- Each of the following is a nasty number
  1. $11(1369453x_{2n+3}-4253x_{2n+4}+132)$
  2. $11(4253x_{2n+2}-13x_{2n+3}+132)$
  3. $3542(y_{2n+4}-102073y_{2n+2}+42504)$

REMARKABLE OBSERVATIONS:

1. Employing the solution (x, y) of (1), each of the following expressions among the special polygonal and pyramidal numbers is a congruent to zero under modulo 11.

$$180\left(\frac{6p^3{}_{x-2}}{pr_{x-2}}\right)^2-\left(\frac{6p^4_{y-1}}{t_{3,2(y-1)}}\right)^2 ;180\left(\frac{2p^5_{x-1}}{t_{4,x-1}}\right)^2-\left(\frac{3p^3{}_y}{t_{3,y+1}}\right)^2$$

2. Employing the linear combinations among the integer solution of (1), one may obtain integer solutions for different geometrical representations. For simplicity and clear understanding, a few illustrations are presented in the table 1 below:

Table 1:

X	Y	Relation between X and Y	Geometrical representation
$(4253x_{n+1}-13x_{n+2})$	$1369453x_{n+1}-13x_{n+3}$	$Y=322X$	Straight line
$(317y_{2n+4}-102073y_{2n+3}+132)$	$(y_{n+2}-317y_{n+1})$	$Y^2=66X$	Parabola
$(4253x_{2n+2}-13x_{2n+3}+132)$	$(1369453x_{n+2}-4253x_{n+3})$	$Y^2=66X$	Parabola
$(317y_{2n+4}-102073y_{2n+3}+132)$	$(4253y_{n+1}-13y_{n+2})$	$Y^2=11880X-3136320$	Parabola
$(y_{n+2}-317y_{n+1})$	$(4253y_{n+1}-13y_{n+2})$	$Y^2=180X^2-3136320$	Hyperbola

3. Using the integer solutions of (1), one may obtain relations among special polygonal numbers. A few relations are exhibited in the following table 2.

Table 2:

m	n	Relations among special polygonal numbers
$\frac{x_{s+1}-1}{2}$	$\frac{y_{s+1}+3}{8}$	$t_{10,n}=90t_{3,m}+10$
$\frac{x_{s+1}-1}{2}$	$\frac{y_{s+1}+4}{9}, s=0,1,2,..$	$t_{20,n}=160t_{3,m}+17$
$\frac{x_{s+1}-1}{2}$	$\frac{y_{s+1}+2}{5}, s=-1,0,1,..$	$t_{12,n}=288t_{3,m}+33$

3. Consider $m=x_{s+1}+y_{s+1}, n=x_{s+1}$. Observe that m>n>0.Treat m,n as the generators of the Pythagorean triangle T(α,β,γ),where $\alpha=2mn,\beta=m^2-n^2,\gamma=m^2+n^2$.

Let A,P represent the area and perimeter of T(α,β,γ)

Then the following interesting relations are observed.

$$\alpha-90\beta+89\gamma=11;91\alpha-\gamma-360\frac{A}{P}=11;x_{s+1}*y_{s+1}=\frac{2A}{P}$$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2=180x^2-11$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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