ISSN: 2456-1312
Volume 1, Number 1
Universal Journal of Mathematics

Hyers - Ulam Stability of a Fredholm Integral Equation with Trigonometric Kernels

K. Ravi, R. Murali, A. Ponmanaselvan

PG & Research Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur – 635 601, Vellore Dist. Tamil Nadu, India.

shckravi@yahoo.co.in, selvaharry@yahoo.com

Abstract

In this paper, authors are interested in proving the Hyers - Ulam stability of a Fredholm integral equation of second kind with the trigonometric kernel function of the form

$$\phi(x) = x + \lambda \int_0^\pi \sin nx \, \sin ns \, \phi(s)\, ds$$

where n is an integer and for all $x \in [0, \pi]$, by using the fixed point method.

Keywords

Hyers - Ulam stability; Fredholm Integral equation of second kind; Fixed Point Method; Kernel.

SUBJECT CLASSIFICATION

Mathematical Subject Classification: 34K20, 26D10, 31K20, 39A10, 34A40, 39B82, 34A80.

INTRODUCTION

An integral equation is an equation in which the unknown function $\phi(x)$ to be determined appear under the intgral sign. A typical form of an integral equation in $\phi(x)$ is of the form

$$\phi(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,s)\, \phi(s)\, ds \qquad (1.1)$$

where $K(x,s)$ is called kernel of the integral equation, $\alpha(x)$ and $\beta(x)$ are the limits of the integration. The standard form of Fredholm linear integral equation of the first kind is of the form

$$f(x) = \int_a^b K(x,s)\, \phi(s)\, ds \qquad (1.2)$$

for all $x \in [a,b]$. The Fredholm integral equation of the second kind has the form

$$\phi(x) = f(x) + \lambda \int_a^b K(x,s)\, \phi(s)\, ds \qquad (1.3)$$

for all $x \in [a,b]$, while its corresponding homogeneous equation is

$$\phi(x) = \int_a^b K(x,s)\, \phi(s)\, ds$$

Integral equations are occur naturally in many fields of science and Engineering. A computational approach to solve integral equation is an essential work in scientific research. The Fredholm integral equation is one of the most important integral equation. Integral equation is encountered in a variety of applications in many fields including Mechanics, Potential theory, Electricity and Magnetism, Kinetic energy of gases, Quantum mechanics, optimization optimal control systems, Communication theory, Mathematical economics, Population genetics, Queuing theory, Mathematical problems of radiative equilibrium, Fluid mechanics, Steady state heat conduction, Fracture mechanics and Radiative heat transfer problems.

In 1940, S. M. Ulam [1] gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. It motivated the study of stability problems for various functional equations. Among the problem raised by S. M. Ulam \cite{SMU}, the following question is concerned about the stability of homomorphisms.

Theorem 1. 1

Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$\rho\left(f(x\,y), f(x)\,f(y)\right) < \delta$$

for all $x, y \in G_1$, then we can find a homomorphism $h : G_1 \to G_2$ exists with

$$\rho\left(f(x), h(x)\right) < \varepsilon$$

for all $x \in G_1$?.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [2] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam's problem for the case of approximately additive mappings, whenG_1 and G_2 are assumed to be Banach spaces. After then, the Hyers - Ulam stability of functional equations [3, 4, 5, 6] and differential equations [7, 8, 9, 10, 11,12] was investigated by several Mathematicians.

Also, In 2015, Z. Gu and J. Huang [13] proved the Hyers - Ulam stability of Fredholm integral equation of form

$$\phi(x) = f(x) + \lambda \int_a^b K(x,s)\,\phi(s)\,ds$$

by Fixed Point method.In 2016, K. Ravi, R. Murali and A. Ponmanaselvan [14] proved the Hyers - Ulam stability of a particular type Fredholm integral equation of second kind with exponential kernel function of the form

$$\phi(x) = 1 + \lambda \int_0^1 e^{x+s}\,\phi(s)\,ds.$$

In this paper, we investigate the Hyers - Ulam stability of Fredholm integral equation of second kind with trigonometric kernel function by using the fixed point method and the successive approximation method in the sense of Z. Gu and J. Huang [13].

PRELIMINARIES

The following Theorems are very useful to prove our main Result.

Theorem 2. 1 (Fixed Point Theorem).

Let (X, ρ) be a complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with $\rho(Tx, Ty) \leq \theta\,\rho(x,y)$, $(0 < \theta < 1)$. Then

(i) there exists an unique fixed point $x^*$of T $(T\,x^* = x^*)$,

(ii) the sequence $\{T^n\,x\}$converges to x^* .

Theorem 2.2(Holder's Inequality).

Assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1, x \in L^p(E)$, $y \in L^q(E)$, then $xy \in L(E)$and

$$\int_E |x(t)\,y(t)|\;dt \leq \left(\int_E |x(t)^p|\right)^{\frac{1}{p}} \left(\int_E |y(t)^q|\right)^{\frac{1}{q}}.$$

Successive Approximation.

Consider the set of successive approximation to the solution $\phi(x)$given by $(1.3)$with the $N\,th$ approximation sum is

$$\phi_N(x) = \sum_{n=0}^{N} \lambda^N\,\phi_n(x),$$

$N = 0, 1, 2, 3, \ldots$. Then the sequence of approximation is generated by an iterative process of successive substitutions on the Fredholm integral equation of second kind is given by

$$\phi_n(x) = \int_a^b K(x,s)\,\phi_{n-1}(s)\,ds.$$

HYERS – ULAM STABILITY

In this section, we investigate the Hyers - Ulam stability of Fredholm integral equation of second kind with sine kernel function by using the Fixed Point Theorem and also by the method of successive approximations in the sense of Z. Gu and J. Huang [13]. That is, if $\phi(x)$ is an approximate solution of

$$\phi(x) = x + \lambda \int_0^\pi \sin nx \, \sin ns \;\; \phi(s)\, ds \,, \qquad (3.1)$$

then there exists an exact solution of the differential equation near to ϕ .

The following Theorem proves the Hyers - Ulam stability of the Fredholm Integral Equation(3.1).

Theorem 3. 1

Suppose that the mapping $\psi : [0,\pi] \to \mathbb{R}$ and the kernel $K(x,s) = \sin nx \, \sin ns \; \in L^2([0,\pi])$. If $\psi(x)$ satisfies the following inequality

$$\left| \psi(x) - x - \lambda \int_0^\pi \sin\, nx \, \sin ns \;\; \psi(s)\, ds \right| \le \varepsilon \qquad (\varepsilon > 0), \qquad (3.2)$$

where

$$\left| \int_0^\pi \sin nx \sin ns \;\; ds \right| \le \frac{1-(-1)^n}{n} \le M = 2\,,$$

for every $x \in [0,\pi]$ and

$$\left| \left\{ \int_0^\pi \int_0^\pi (\sin nx \;\; \sin ns\,)^2 \; dx \; ds \right\}^{\frac{1}{2}} \right| \le M = 2,$$

for $n \neq 0 \; and \; 1$,then there exists a solution ϕ satisfies (3.1), and

$$|\, \phi(x) - \psi(x)| < \frac{1}{1-2\lambda}\, \varepsilon$$

for $-\infty < \lambda < \frac{1}{2}$and for every $x \in [0,\pi]$.

Proof. First of all, we define an operator T by

$$(T\,\phi)(x) = x + g(x) + \lambda \int_0^\pi \sin nx \, \sin ns \; \phi(s)\; ds, \qquad \phi \in L^2([0,\pi]). \qquad (3.3)$$

Since we have,

$$\left| \int_0^\pi \sin nx \sin ns \;\; ds \right| = \left| \sin nx \int_0^\pi \sin ns \;\; ds \right| \le \frac{1-(-1)^n}{n} < 2,$$

for $n \neq 0 \; and \; 1$, and also we have,

$$\left| \left\{ \int_0^\pi \int_0^\pi (\sin nx \;\; \sin ns\,)^2 \; dx \; ds \right\}^{\frac{1}{2}} \right| \le M = 2.$$

Now, we define a metric ρ by,

$$\rho(\varphi_1, \varphi_2) = \left\{ \left\{ \int_0^\pi |\varphi_1(x) - \varphi_2(x)|^2 \, dx \right\}^{\frac{1}{2}} : \; \varphi_1, \varphi_2 L^2([0,\pi]) \right\}.$$

We have,

$$\int_0^\pi \left| \int_0^\pi \sin nx \sin ns \;\; \phi(s)\, ds \right|^2 \; dx \le \int_0^\pi \left\{ \int_0^\pi \{\sin nx \; \sin ns\,\}^2 \; ds \int_0^\pi \phi^2(s)\, ds \right\} dx$$

$$\leq \int_0^\pi \phi^2(s)\, ds \; . \int_0^\pi \left\{ \int_0^\pi \{\sin nx \, \sin ns\}^2 \, ds \right\} dx \;\; \leq \infty .$$

Which implies that $T\phi \in L^2([0,\pi])$ and T is a self - mapping of $L^2([0,\pi])$. Thus, the solution of equation (3.3) is the fixed point of T .Moreover,

$$\rho(T\varphi_1, T\varphi_2) = \left\{ \int_0^\pi |(T\varphi_1)(x) - (T\varphi_2)(x)|^2 \, dx \right\}^{\frac{1}{2}}$$

$$\leq \left\{ \int_0^\pi \left| \lambda \int_0^\pi \sin nx \, \sin ns \, \big(\varphi_1(s) - \varphi_2(s)\big) \, ds \right|^2 dx \right\}^{\frac{1}{2}}$$

$$\leq |\lambda| \left\{ \int_0^\pi \left\{ \int_0^\pi (\sin nx \, \sin ns)^2 \, ds \int_0^\pi |\varphi_1(s) - \varphi_2(s)|^2 \, ds \right\} dx \right\}^{\frac{1}{2}}$$

$$\leq |\lambda| \left\{ \int_0^\pi \int_0^\pi (\sin nx \, \sin ns)^2 \, ds \, dx \right\}^{\frac{1}{2}} \quad \rho(\varphi_1, \varphi_2) .$$

And we note that, for $n \neq 0 \; and \; 1$, and all $n \in \mathbb{z}$, we have

$$\left| \left\{ \int_0^\pi \int_0^\pi (\sin nx \;\; \sin ns)^2 \, dx \, ds \right\}^{\frac{1}{2}} \right| \leq M = 2.$$

Thus we prove, T is a contractive operator. It follows from Theorem 2. 1 that equation $(3.3\,)$ has a unique solution $\phi^* \in L^2([0,\pi])$ where $\phi^* = \lim_{r\to\infty} \phi_r$ for

$$\phi_r(x) = x + g(x) + \lambda \int_0^\pi \sin nx \, \sin ns \, \phi_{r-1}(s) \, ds$$

and $\phi_0 \in L^2([0,\pi])$ is an arbitrary function.

Now, assume that $g(x) = 0$ in equation (3.3), then we know that there exists an unique solution $\phi^* \in L^2([0,\pi])$ of

$$\phi(x) = x + \lambda \int_0^\pi \sin nx \, \sin ns \;\; \phi(s) \, ds \, , \qquad (3.4)$$

where $\phi^* = \lim_{r\to\infty} \phi_r$ for

$$\phi_r(x) = x + \lambda \int_0^\pi \sin nx \, \sin ns \, \phi_{r-1}(s) \, ds$$

and $\phi_0 \in L^2([0,\pi])$ is an arbitrary function. Then, let $\psi \in L^2([0,\pi])$ be a solution of Inequality (3.2) and

$$\psi(x) - x - \lambda \int_0^\pi \sin \, nx \, \sin ns \;\; \psi(s) \, ds \;\; =: \, h(x). \qquad (3.5)$$

Obviously, we have $|h(x)| \leq \varepsilon \;\; for \; all \;\; x \in [0,\pi]$. Then we can know that the solution of equation (3.5) is $\psi^* \in L^2([0,\pi])$ of

$$\psi(x) = h(x) + x + \lambda \int_0^\pi \sin \, nx \, \sin ns \;\; \psi(s) \, ds$$

Where,

$$\psi^*(x) = \lim_{r\to\infty} \psi_r \, (x)$$

for

$$\psi(x) = x + h(x) + \lambda \int_0^\pi \sin nx \, \sin ns \, \psi_{r-1}(s) \, ds$$

and $\psi_0 \in L^2([0,\pi])$is an arbitrary function.At last, let$\psi_0(x) = \phi_0(x) = 0$, then we have

$$|\phi_1(x) - \psi_1(x)| = |h(x)| \leq \varepsilon$$

$$|\phi_2(x) - \psi_2(x)| = \left| h(x) + \lambda \int_0^\pi \sin nx \sin ns \left(\phi_1(x) - \psi_1(x)\right) ds \right| \leq \varepsilon \left(1 + \lambda \int_0^\pi |\sin nx \sin ns| \, ds\right)$$

$$|\phi_3(x) - \psi_3(x)| = \left| h(x) + \lambda \int_0^\pi \sin nx \sin ns \left(\phi_2(x) - \psi_2(x)\right) ds \right|$$

$$\leq \varepsilon + \varepsilon \lambda \int_0^\pi |\sin nx \sin ns_2| \left(1 + \lambda \int_0^\pi |\sin ns_2 \sin ns_1| \, ds_1\right) ds_2$$

$$\leq \varepsilon \left(1 + \lambda \int_0^\pi |\sin nx \sin ns| \, ds + \lambda^2 \int_0^\pi |\sin nx \sin ns_2| \int_0^\pi |\sin ns_2 \sin ns_1| \, ds_1 \, ds_2\right)$$

... ...

$$|\phi_r(x) - \psi_r(x)| = \left| h(x) + \lambda \int_0^\pi \sin nx \sin ns \left(\phi_{r-1}(x) - \psi_{r-1}(x)\right) ds \right|$$

$$\leq \varepsilon \left(1 + \lambda \int_0^\pi |\sin nx \sin ns| \, ds + \lambda^2 \int_0^\pi |\sin nx \sin ns_2| \int_0^\pi |\sin ns_2 \sin ns_1| \, ds_1 \, ds_2 + \cdots \right.$$
$$\left. + \lambda^{r-1} \int_0^\pi |\sin nx \sin ns_{r-1}| \ldots \int_0^\pi |\sin ns_2 \sin ns_1| \, ds_{r-1} \ldots ds_1\right)$$

$$\leq \varepsilon \left(1 + M\lambda + (M\lambda)^2 + (M\lambda)^3 + \cdots + (M\lambda)^{r-1}\right)$$

$$\leq \varepsilon \left(\frac{1 - (M\lambda)^r}{1 - M\lambda}\right)$$

$$|\phi^*(x) - \psi^*(x)| \leq \frac{\varepsilon}{1 - M\lambda} = \frac{\varepsilon}{1 - 2\lambda},$$

As $r \to \infty$ and $-\infty < \lambda < \frac{1}{2}$. Hence ϕ has an exact solution near to it. Thus, the proof is completed.

CONCLUSION

We have established the Hyers - Ulam stability of a Fredholm Integral equation of second kind with Trigonometric kernels in the sense of Z. Gu and J. Huang. This result will serve as an illustrative example to apply the Hyers - Ulam stability to integral equations. This study will help the readers to apply similar type of problems in integral equations.

REFERENCES

[1] S. M. Ulam, A Collection of Mathematical Problems, Interscience Publ., New York, 1960.

[2] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A 27 (1941) 222 - 224.

[3] T. Aoki, On the stability of the linear transformation in Banach Spaces, J. Math. Soc.Japan 2 (1950), 64 - 66.

[4] Y. Li and L. Hua, Hyers - Ulam stability of Polynomial equation, Banach journal of Mathematical Analysis, Vol. 3, no. 2, (2009), 86 - 90.

[5] Th. M. Rassias, On the stability of the linear mappings in Banach Spaces, Proc. Amer. Math. Soc. 72 (1978), 297 - 300.

[6] Th. M. Rassias, On the stability of functional equations and a problem of Ulam, Acta. Appl. Math. 62 (2000) 23 - 130.

[7] M. Obloza, Hyers stability of the linear differential equation, Rocznik Nauk - Dydakt. Prace Math., 13, pp. 259 - 270, 1993.

[8] M. Obloza, Connection between Hyers and Lyapunov stability of the ordinary differential equations, Rocznik Nauk - Dydakt. Prace., Mat. 14, pp. 141 -146, 1997.

[9] C. Alsina, R. Ger, On Some inequalities and stability results related to the exponential function, Journal of Inequalities Appl. 2 (1998) 373 – 380.

[10] S. E. Takahasi, T. Miura, S. Miyajima, On the Hyers - Ulam stability of the Banach space - valued differential equation $y^{'} = \lambda y$, Bull. Korean Math. Soc. 39 (2002) 309 - 315.

[11] S. M. Jung, Hyers - Ulam stability of linear differential equations of first order (II), Appl. Math. Lett. 19 (2006), 854 - 858.

[12] I. A. Rus, Ulam Stabilities of Ordinary Differential Equations in a Banach Space, Carpathian J. Math. 26 (2010), No.1, 103 - 107.

[13] Z. Gu and J. Huang, Hyers - Ulam stability of Fredholm Integral equation, Mathematica Aeterna, Vol. 5, 2015, no. 2, 257 - 261.

[14] K. Ravi, R. Murali and A. Ponmanaselvan, Stability of a particular Fredholm Integral equation, Asian Journal of Mathematics and Computer Research, International Knowledge Press, Vol. 11, No. 4, pp. 325 – 333, 2016.