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ABSOLUTE ALMOST CONVERGENCE OF FOURIER SERIES AND CONJUGATE SERIES

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Abstract:

The main object of the present paper is to study the absolute almost convergence of Fourier Series and Conjugate Series using fractional means of generating function and extend a recent result of Das and Ray.

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1 Introduction

Given an intimate series $\sum_{n=0}^{\infty} a_n$ which we shall denote by *a*, we write $S_n = \sum_{k=0}^{n} a_k$. If

$$t_{m,n} = \frac{1}{m+1} \sum_{k=0}^{m} S_{n+k} \rightarrow s \text{ uniformly in } n \tag{1.1}$$

Then $\sum a_n$ is said to be almost convergent to *s* (see [8]). Let $\hat{c}$ denote the set of almost convergent sequences. The series a (or the sequence (s_n) is said to be absolutely almost convergent (see[3],[4],[5],[6]) if

$$\sum_{m=0}^{\infty} \left|\phi_{m,n}(a)\right| < \infty \text{ uniformly in } n. \tag{1.2}$$

where

$$\phi_{mn} = a_0 = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu a_{n+\nu}, \ (n \geq 1)$$

$$\phi_{0,n} = a_0 \tag{1.3}$$

We denote absolutely almost convergent sequences by the symbol $\hat{l}$. An infinite series $\sum a_n$ is absolutely $(C,\alpha), \alpha > 0$ summable $\left(|C,\alpha|, \, in \ short\right)$ if

$$\sum_{n=1}^{\infty} \frac{\left|\tau_n^{\alpha}\right|}{n} < \infty$$

where τ_n is the (C,α) mean ([1]) of the sequence $\{n, a_n\}$; that is,

$$\tau_n = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} k a_k$$

and where the co-efficents A_n^{α} are given by

$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^{\alpha} x^n, \; |x| < 1.$$

Let *l* and $\hat{l}$ denote the set of absolutely convergent series and absolutely almost convergent series, then the following results are known: (i) $l \subset \hat{l} \subset |C,1|$ (ii) $\hat{l}$ and $|C,\alpha|, 0<\alpha<1$ are mutually exclusive [4].

Let *f* be a 2π -periodic function and Lebesgue integrable over $(-\pi,\pi)$. The Fourier Series of *f* at *x* is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

The series conjugate to Fourier Series is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x).$$

We write,

$$\phi(t) = \phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,\phi(u)\,du,\; \alpha>0,\; \Phi_0(t)=\phi(t)$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\;\Phi_\alpha(t), \alpha \ge 0$$

$\psi_\alpha(t)$ and $\Psi_\alpha(t)$ are defined in a similar way.

2 Main Theorems

Bosonquet first studied in the absolute Cesaro summability of Fourier Series and his result reads as follows:

Theorem A[1] $\phi_\alpha(t) \in BV(0,\pi) \Rightarrow \sum A_n(x) \in |C,\beta|,\; \beta > \alpha \ge 0$

Recently Das and Ray proved the following.

Theorem B [7] $\phi(t) \in BV(0,\pi) \Rightarrow \sum A_n(x) \in \hat{l}$.

We now extend Theorem B and prove the following theorem.

Theorem 1 Let $0<\alpha<1$. Then

$$\phi_\alpha(t) \in BV(0,\pi) \Rightarrow \sum A_n(x) \in \hat{l}\,,$$

The result is not necessarily true if $\alpha = 1$.

We may remark that the result of Theorem 1 is significant in view of the fact that the $\hat{l}$ and $|C,\alpha|, 0<\alpha<1$ are not comparable.

With regard to the Conjugate Series we prove.

Theorem 2 Let $0<\alpha<1$. Then

$$\int_0^{\pi} u^{-\alpha} \left| d\Psi_\alpha(u) \right| < \infty \rightarrow \sum_{n=0}^{\infty} B_n(x) \in \hat{l} .$$

The result is not necessarily true for $\alpha = 1$.

3 Lemmas

We need the following additional notations and lemmas.

$$T_m(n) = \sum_{\nu=1}^{m} \nu A_{n+\nu}(x) \tag{3.1}$$

$$l_m(n,t) = \sum_{\nu=1}^{m} \frac{\nu}{n+\nu} \sin(n+\nu)t \tag{3.2}$$

$$R_m(n,t) = \sum_{\nu=1}^{m} \nu \cos(n+\nu)t \tag{3.3}$$

$$K_m(n,t) = \sum_{\nu=1}^{n} \nu \cos(n+\nu)t \tag{3.4}$$

$$J(m,n,u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha} R_m(n,t)\,dt \tag{3.5}$$

$$V(m,n,u) = \frac{1}{\Gamma(\alpha+1)} \int_0^{u} \nu^{\alpha} \frac{d}{d\nu} J(m,n,\nu)\,d\nu \tag{3.6}$$

Lemma 1 Uniformly in *n* and $0 < t \le \pi$

(i) $$l_m(n,t) = \begin{cases} O(m) \\ O(t^{-1}) \end{cases}$$

(ii) $$R_m(n,t) = \begin{cases} O(m^2) \\ O(mt^{-1}) \end{cases}$$

where $l_m(n,t)$ and $R_m(n,t)$ are defined in (3.2) and (3.3) respectively.

Proof: As $\left| \frac{\nu}{n+\nu} \sin(n+\nu)t \right| \le 1$ uniformly in *n*, the first estimate follows at once. Next, as $\frac{\nu}{n+\nu}$ is monotonic decreasing in ν, we have uniformly in *n*

$$\left| l_m(n,t) \right| \le \frac{m}{n+m} \max_{M \le \nu \le M'} \left| \sum_{M}^{M'} \sin(n+\nu)t \right| = O(t^{-1}).$$

We omit the proof of (ii) as it is similar to that of (i).

Lemma 2 uniformly in *n* and for $0 < u \leq \pi$

(i) $$J(m,n,u) = O\left(m^{1+\alpha}\right)$$

(ii) $$J(m,n,u) = O\left(m^{\alpha} u^{-1}\right)$$

Proof: We write

$$\Gamma(1-\alpha) J(m,n,u) = \left(\int_{u}^{u+\frac{1}{m}} + \int_{u+\frac{1}{m}}^{\pi} \right) (t-u)^{-\alpha} R_m(n,t)\, dt$$

$$= I_1 + I_2, \text{ say} \qquad (3.7)$$

Using Lemma 1(ii)

$$I_1 = O\left(m^2\right) \int_{u}^{u+\frac{1}{m}} (t-u)^{-\alpha}\, dt$$

$$= O\left(m^2\right) O\left(m^{\alpha-1}\right) = O\left(m^{1+\alpha}\right).$$

By second Mean value Theorem followed by an application of Lemma 1(i), we get, for $u + \frac{1}{m} < \pi' < \pi$.

$$I_2 = \int_{u+\frac{1}{m}}^{\pi} (t-u)^{-\alpha} R_m(n,t)\, dt$$

$$= m^{\alpha} \int_{u+\frac{1}{m}}^{\pi'} R_m(n,t) dt$$

$$= m^{\alpha} \left[l_m\left((n,\pi') - l_m\left(n, u + \frac{1}{m} \right) \right) \right]$$

$$= m^{\alpha} O(m) = O\left(m^{1+\alpha}\right)$$

and this ensures Lemma 2(i). Using Lemma 1(ii)

$$I_1 = \int_{u}^{u+\frac{1}{m}} (t-u)^{-\alpha} R_m(n,t) dt = O(m) \int_{u}^{u+\frac{1}{m}} t^{-1} (t-u)^{-\alpha}\, dt$$

$$O(m) \int_{u}^{u+\frac{1}{m}} (t-u)^{-\alpha}\, dt = O\left(m^{\alpha} u^{-1}\right).$$

from the proof of the Lemma 2(i), we have

$$I_2 = m^\alpha \left[l_m \left(n, \pi' - l_m \left(n, u + \frac{1}{m} \right) \right) \right] = O\left(m^\alpha u^{-1}\right)$$

Using Lemma 1(i); and this completes the proof of lemma 2(ii).

Lemma 3 Uniformly in *n*

(i) $V(m,n,u) = O\left(u^\alpha m^{1+\alpha}\right)$

(ii) $V(m,n,u) = O\left(m^\alpha u^{\alpha-1}\right).$

Proof: By integration by parts and Lemma 2

$$\Gamma(\alpha+1)V(m,n,u) = \int_0^u v^\alpha \frac{d}{dv} J(m,n,u)\, dv$$

$$= u^\alpha O\left(m^{1+\alpha}\right) + O\left(m^{1+\alpha}\right) \int_o^u v^{\alpha-1}\, dv = O\left(u^\alpha m^{1+\alpha}\right).$$

This established Lemma 3(i). As $V(m,n,u) = 0$, we have

$$\Gamma(\alpha+1)V(m,n,u) = \Gamma(\alpha+1)V(m,n,\pi) - \int_u^\pi v^\alpha \frac{d}{dv} J(m,n,u)\, dv$$

$$= -\int_u^\pi v^\alpha \frac{d}{dv} J(m,n,v)\, dv$$

$$= -\left[v^\alpha J(m,n,u)\right]_{v=u}^{\pi} + \alpha \int_u^\pi v^{\alpha-1} J(m,n,v) dv$$

$$= u^\alpha J(m,n,u) + \alpha \int_u^\pi v^{\alpha-1} J(m,n,v) dv$$

$$= u^\alpha O\left(m^\alpha u^{-1}\right) + O(1) \int_u^\pi v^{\alpha-1} m^{\alpha-1} v dv$$

$$= O\left(m^\alpha u^{\alpha-1}\right) + O\left(m^\alpha\right) \int_u^\pi v^{\alpha-2} dv \ (as\ \ 0<\alpha<1)$$

$$= O\left(m^\alpha u^{\alpha-1}\right)$$

by using the estimate of Lemma 2(ii).

This completes the proof of Lemma 3(ii).

4 Proof of Theorem 1 As ([4], page 51)

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt\ dt,$$

we have,

$$T_m(n) = \sum_{\nu=1}^{m} \nu A_{n+\nu}(x)$$

$$= \frac{2}{\pi}\int_0^{\pi} \phi(t)\, K_m(n,t)dt \qquad (4.1)$$

where $K_m(n,t)$ is defined in (3.4). Now using the inversion formula(see[1])

$$\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\, d\Phi_\alpha(u),\ 0<\alpha<1 \qquad (4.2)$$

We get from (4.1)

$$T_m(n) = \frac{2}{\pi\Gamma(1-\alpha)}\int_0^{\pi} R_m(n,t)\, dt \int_0^t (t-u)^{-\alpha}\, d\Phi_\alpha(u)$$

$$= \frac{2}{\pi\Gamma(1-\alpha)}\int_0^{\pi} d\Phi_\alpha(u)\int_u^{\pi}(t-u)^{-\alpha} R_m(n,t)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} J(m,n,u)\, d\Phi_\alpha(u).$$

Now by integration by parts

$$T_m(n) = \frac{2}{\pi}[\Phi_\alpha(u)J(m,n,u)]_{u=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi}\Phi_\alpha(u)\frac{d}{dt}\, J(m,n,u)\, du$$

$$= -\frac{2}{\pi}\int_0^{\pi}\Phi_\alpha(u)\frac{d}{du}\, J(m,n,u)du\ (\text{since}\ \Phi_\alpha(0)=0, J(m,n,\pi)=0)$$

$$= -\frac{2}{\pi\Gamma(\alpha+1)}\int_0^{\pi} u^{\alpha}\phi_\alpha(u)\, J(m,n,u)du$$

$$= -\frac{2}{\pi}\left[\phi_\alpha(u)\left(\int_0^u \frac{v^{\alpha}}{\Gamma(\alpha+1)}\frac{d}{dv}\, J(m,n,v)\, dv\right)\right]_{u=0}^{\pi}$$

$$+ \frac{2}{\pi}\int_0^{\pi} d\phi_\alpha(u)\left(\frac{1}{\Gamma(\alpha+1)}\int_0^u v^{\alpha}\frac{d}{dv}\, J(m,n,v)\, dv\right)$$

$$= -\frac{2}{\pi}\left[\phi_\alpha(u)V(m,n,\pi) + \frac{2}{\pi}\int_0^{\pi} V(m,n,u)\, d\phi_\alpha(u)\right] \qquad (4.3)$$

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In the special case when $\phi(t)=1$, we have $\phi_\alpha(t)=1, T_m(n)=0$ and the last integral in (4.3) vanishes. Hence (putting $\phi(t)=1, V(m,n,\pi)=0$. Thus(4.3) reduces to

$$T_m(n)=\frac{2}{\pi}\int_0^\pi V(m,n,u)\,d\phi_\alpha(u) \qquad (4.4)$$

Now the series $\sum A_n(x)\in\hat{l}$ if and only if

$$\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\left|\frac{2}{\pi}\int_0^\pi V(m,n,u)\,d\phi_\alpha(u)\right|<\infty \qquad (4.5)$$

uniformly in *n*..

As $\int_0^\pi |d\phi_\alpha(u)|$ is infinite, for the validity of (4.5), it is enough to show that, uniformly in $0<u\le\pi$ and uniformly in *n*.

$$\sum\equiv\sum_{m=1}^{\infty}\frac{1}{m(m+1)}|V(m,n,u)|=O(1) \qquad (4.6)$$

we write

$$\sum\equiv\left(\sum_{m\le u^{-1}}+\sum_{m>u^{-1}}\right)|V(m,n,u)|=\sum_1+\sum_2 \quad \text{say} \qquad (4.7)$$

using Lemma 3(i)

$$\sum_1=O(u^{-\alpha})\sum_{m\le u^{-1}}\frac{m^{1+\alpha}}{m(1+m)}=O(u^\alpha)\sum_{m\le u^{-1}}m^{\alpha-1}=O(1) \qquad (4.8)$$

uniformly in *n*. Using Lemma 3(ii)

$$\sum_2=\sum_{m>u^{-1}}\frac{1}{m(m+1)}O(m^\alpha u^{\alpha-1})$$

$$=O(u^{\alpha-1})\sum_{m>u^{-1}}\frac{1}{m^{2-\alpha}}=O(u^{\alpha-1})O(u^{1-\alpha})=O(1) \qquad (4.9)$$

Using (4.8) and (4.9), we get (4.6) and this completes the proof of first part of Theorem 1.

The absolute almost convergence of Fourier Series is a non-local property of its generating function as $\hat{l}\subset|C,1|$ and it is known [1] that $[C,1]$ summability of fourier series can not be ensured by local condition. Hence Theorem 1 breaks down when $\alpha=1$ as the hypothesis $\phi_1(t)\in BV(0,\pi)$ is a local condition [1] although it appears to be non-local.

5 Further Lemmas

We need the following additional notations and Lemmas for the proof of Theorem 2.

$$\overline{T}_m(n)=\sum_{\nu=1}^{m}\nu\,B_{n+\nu}(x) \qquad (5.1)$$

$$\tilde{l}_m(n,t) = \sum_{\nu=1}^{m} \frac{\nu}{n+\nu} \cos(n+\nu)t \tag{5.2}$$

$$\tilde{R}_m(n,t) = -\sum_{\nu=1}^{m} \nu \sin(n+\nu)\, t \tag{5.3}$$

$$\tilde{J}(m,n,u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha}\, \tilde{R}_m(n,t)\, dt \tag{5.4}$$

Lemma 4 Uniformly in *n* and $0 < t < \pi$

(i) $$\tilde{l}_m(n,t) = \begin{cases} O(m) \\ O(t^{-1}) \end{cases}$$

(ii) $$\tilde{R}_m(n,t) = \begin{cases} O(m^2) \\ O(nt^{-1}) \end{cases}$$

Lemma 5 Uniformly in *n* and $0 < u < \pi$

(i) $$\tilde{J}(m,n,u) = O(m^{1+\alpha})$$

(ii) $$\tilde{J}(m,n,u) = O(m^{\alpha} u^{-1})$$

We omit the proof of Lemma 5 it is similar to the proof of Lemma 3.

6 Proof of Theorem 2

We have,

$$\tilde{T}_m(n) = \sum_{\nu=1}^{m} \nu\, B_{n+\nu}(x)$$

$$= -\frac{2}{\pi} \sum_{\nu=1}^{m} \nu \int_0^{\pi} \psi(t) \sin(n+\nu)t\, dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} \psi(t) \left(\sum_{\nu=1}^{m} \nu \sin(n+\nu)\, t \right) dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \psi(t)\, \tilde{R}_m(n,t)\, dt \tag{6.1}$$

It is known [2] that, for $0 < \alpha < 1$

$$\psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t} (t-u)^{-\alpha} d\Psi_\alpha(u)$$

and hence from (6.1), we get

$$\tilde{T}_m(n) = \frac{2}{\pi\Gamma(1-\alpha)} \int_0^{\pi} \tilde{R}_m(n,t)\,dt \int_0^{t} (t-u)^{-\alpha}\; d\Psi_\alpha(u)$$

$$= \frac{2}{\pi\Gamma(1-\alpha)} \int_0^{\pi} d\Psi_\alpha(u) \int_0^{\pi} (t-u)^{-\alpha}\; \tilde{R}_m(n,t)\,dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \tilde{J}(m,n,u)\, d\Psi_\alpha(u) \qquad (6.2)$$

where $\tilde{J}(m,n,u)$ is defined in (5.4).

Hence $\sum B_n(x) \in \hat{l}$ if and only if, uniformly in n

$$\sum_{m=1}^{\infty} \frac{\left|\tilde{T}_m(n)\right|}{m(m+1)} < \infty \;;$$

That is, uniformly in n

$$\sum_{m=1}^{\infty} \frac{1}{m(m+1)} \left| \frac{2}{\pi} \int_0^{\pi} \tilde{J}(m,n,u)\, d\Psi_\alpha(u) \right| < \infty;$$

By the hypothesis $\int_0^{\pi} u^{-\alpha} \left|d\Psi_\alpha(u)\right| < \infty$ and hence it remains to show that uniformly in $0 < u \le \pi$ and uniformly in n

$$\sum^{*} \equiv \sum_{m=1}^{\infty} \frac{\left|\tilde{J}(m,n,u)\right|}{m(m+1)} = O\left(u^{-\alpha}\right) \qquad (6.3)$$

writing

$$\sum^{*} = \left(\sum_{m \le u^{-1}} + \sum_{m > u^{-1}} \right) \frac{\tilde{J}(m,n,u)}{m(m+1)}$$

and using Lemma 5(i) and Lemma 5(ii) respectively over the sums $\sum_{m \le u^{-1}}$ and $\sum_{m > u^{-1}}$, we obtain (6.3) and this completes the proof of Theorem 2.

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