ISSN 2456-1312
Volume 2, Number 1

Universal Journal of Mathematics

An Application of improvement of New Homotopy Perturbation Method for Solving Third Order Nonlinear Singular Partial Differential Equations

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Abstract:

In this paper, the new homotopy perturbation method (NHPM), is applied to solve third order nonlinear singular partial differential equations. Four examples are presented to illustrate the method, the (NHPM) is powerful and capable method to solve linear and nonlinear problems directly.

Keywords:

New homotopy perturbation method; Blasius equations; nonlinear partial differential equations.

2010 Mathematics Subject Classification: 35E15, 14F35 ,35A35

(1) Introduction:

In recent years, much attention has been given to develop some analytical methods for solving integral equations including the perturbation methods and decomposition method. We study the problems of partial differential equations (PDEs), such systems arise in many areas of mathematics, engineering and physical sciences. These equations are often too complicated to be solved exactly and even if an exact solution is obtained, the required calculations may be too complicated. Very recently, many powerful methods have been presented. Approximate analytical schemes as the variational iteration method (VIM) (He, 1999) and homotopy perturbation method (HPM) (He, 1999) have been very widely used to solve partial differential equations (PDEs) for many applications in science and engineering.

In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Many powerful methods have been presented, such as the new homotopy perturbation method and standard (HPM) for solving partial differential equations [1-3,5,7-11], the modified decomposition method and Adomian's decomposition method [4,13-14,16]. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [17-19,22], and many others [6,9,12,21], the aim of this work is to employ (NHPM) to obtain the exact solution and approximate solution of the systems of nonlinear partial differential equations. The rest of this work is organized as follows:

In section 2 gives an analysis of the method (NHPM). In section 3 four examples are given to illustrate the proposed approach, In section 4 is reserved for conclusions.

In the next section we discuss the analysis of our method (NHPM).

(2) Analysis of method (NHPM):

The main aim of this section is to know how to solve the nonlinear partial differential equations by using the new homotopy perturbation method (NHPM).

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation,

$$A\left(u(x)\right)-f\left(r(x)\right)=0,\qquad r(x)\in\Omega \tag{1}$$

with the boundary conditions,

$$B\left(u(x),\frac{\partial u(x)}{\partial n}\right)=0,\qquad r(x)\in\Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r(x)),$ is a known analytical function, and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator. Therefore, Eq. (1) can be rewritten as:

$$L(u(x)) + N(u(x)) - f(r(x)) = 0, \tag{3}$$

By the homotopy technique, we construct a homotopy $U(r(x), p) : \Omega \times [0,1] \to \mathbb{R}$, which satisfies,

$$\begin{aligned} H\left(U(x), p\right) &= (1-p)\left[L(u(x)) - u_0(x)\right] \\ &+ p\left[A(U(x)) - f(r(x))\right] = 0, \quad p \in [0,1], \qquad r(x) \in \Omega, \end{aligned} \tag{4}$$

or

$$\begin{aligned} H\left(U(x), p\right) &= L(U(x)) - L(u_0(x)) + pL(u_0(x)) \\ &+ p\left[N(U(x)) - f(r(x))\right] = 0 \end{aligned} \tag{5}$$

where $p \in [0,1]$, the homotopy parameter p, always changes from zero to unity. In case $p = 0$, the Eq. (4) and Eq. (5), becomes the linearized equation.

$$\begin{aligned} H\left(U(x), 0\right) &= L(U(x)) - L(u_0(x)), \\ H\left(U(x), 0\right) &= L(U(x)) + N(U(x)) - f(r(x)) \end{aligned} \tag{6}$$

and when $p = 1$, the Eq. (4) or Eq. (5) turns out to be the original differential equation (1).

$$\begin{aligned} H\left(U(x), 1\right) &= A(U(x)) - f(r(x)) = 0, \\ H\left(U(x), 1\right) &= L(U(x)) + N(U(x)) - f(r(x)) = 0 \end{aligned} \tag{7}$$

According to the (HPM), we can first use the embedding parameter p as a small parameter, and assume that the solutions of Eqs. (4) and Eq. (5) can be represented as a power series in p as:

$$U(x) = U_0(x) + pU_1(x) + p^2U_2(x) + p^3U_3(x) + \cdots, \tag{8a}$$

The approximate solution of Eq. (1), therefore, can be readily obtained,

$$U(x) = \lim_{p \to 1} U(x) = U_0(x) + U_1(x) + U_2(x) + U_3(x) + \cdots, \tag{8b}$$

The convergence of the series (8b) has been proved in [1,22].

Now let us write the Eq. (5) in the following form:

$$L(U(x)) = u_0(x) + p\left[f(r(x)) - u_0(x) - N(U(x))\right], \tag{9}$$

By applying the inverse operator L^{-1}, to both sides of Eq. (9), we obtain:

$$\begin{aligned} U(x) &= L^{-1}\left(u_0(x)\right) + \\ & p\left[L^{-1}\left(f(r(x))\right) - L^{-1}\left(u_0(x)\right) - L^{-1}\left(N(U(x))\right)\right], \end{aligned} \tag{10}$$

Suppose that the initial approximation of Eq. (1) given by:

$$u_0(x)=\sum_{n=0}^{\infty}\alpha_n P_n(x), \tag{11}$$

where $a_0, a_1, a_2, \ldots$ are unknown coefficients and $P_0(x), P_1(x)_1, P_2(x), \ldots$ are specific functions depending on the problem.

Substituting Eqs. (8) and Eq. (11) into the Eq. (10), we get,

$$\sum_{n=0}^{\infty} p^n U_n(x) = L^{-1}\left(\sum_{n=0}^{\infty}\alpha_n P_n(x)\right) + p\left[L^{-1}\left(f(r(x))\right) - L^{-1}\left(\sum_{n=0}^{\infty}\alpha_n P_n(x)\right) - L^{-1}N\left(\sum_{n=0}^{\infty} p^n U_n(x)\right)\right]. \tag{12}$$

Comparing coefficients of terms with identical powers of $p,$ leads to,

$$\left.\begin{array}{l} p^0 : U_n(x) = L^{-1}\left(\sum_{n=0}^{\infty}\alpha_n P_n(x)\right), \\ p^1 : U_1(x) = L^{-1}\left(f(r(x))\right) - L^{-1}\left(\sum_{n=0}^{\infty}\alpha_n P_n(x)\right) - L^{-1}N\left(\sum_{n=0}^{\infty} p^n U_n(x)\right), \\ p^2 : U_2(x) = -L^{-1}N\left(U_0(x), U_1(x)\right), \\ \vdots \\ p^j : U_j(x) = -L^{-1}N\left(U_0(x), U_1(x), \ldots, U_{j-1}(x)\right) \\ \vdots \end{array}\right\}, \tag{13}$$

Now if we went to solve the above equations in such away that $U_1(x)=0,$ in some cases we set the Taylor series of $U_1(x)$ equal to zero, then Eq. (13) result is; $U_2(x) = U_3(x) = \cdots = U_j(x) = \cdots = 0.$

Therefore, the exact solution obtained by using,

$$u(x) = U_0(x) = L^{-1}\left(\sum_{n=0}^{\infty}\alpha_n P_n(x)\right). \tag{14}$$

Where $N = \sum_{m=0}^{\infty} A_m(x),$ are nonlinear operator, then A_m 's, called the Adomian polynomials, are defined as [4],

$$A_m = \left[\frac{1}{m!}\frac{d^m}{d\lambda^m} N\left(\sum_{i=0}^{m}\lambda^i u_i\right)\right]_{\lambda=0}, \tag{15}$$

To show the capability of the method (NHPM) applied to some examples in the next section.

(3) Applications of the method (NHPM):

In this section, to illustrate the method and to show the ability of the method four examples are presented.

Example (3.1):

We consider the following singular nonlinear initial value problem (IVP) of third order partial differential equations see [12],

$$\frac{\partial^3 u(x,t)}{\partial t^3}+\frac{4}{t}\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{2}{t^2}\frac{\partial u(x,t)}{\partial t}=\frac{\partial^3 u(x,t)}{\partial t^3}\cdot\frac{\partial u(x,t)}{\partial x}+\frac{x}{t^2}+\frac{12}{t}, \tag{16}$$

subject to the initial conditions,

$$u(x,0)=0, \qquad u_t(x,0)=\frac{x}{2}, \qquad u_{tt}(x,0)=2, \tag{17}$$

To solve Eq.(16) by using (NHPM), we construct the following homotopy:

$$(1-p)\left(\frac{\partial^3 U(x,t)}{\partial t^3}-u_0(x,t)\right)$$

$$+p\left(\begin{array}{l}\frac{\partial^3 U(x,t)}{\partial t^3}+\frac{4}{t}\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{2}{t^2}\frac{\partial U(x,t)}{\partial t}\\ -\frac{\partial^3 U(x,t)}{\partial t^3}\cdot\frac{\partial U(x,t)}{\partial x}-\frac{x}{t^2}-\frac{12}{t}\end{array}\right)=0, \tag{18}$$

or

$$\frac{\partial^3 U(x,t)}{\partial t^3}=u_0(x,t)$$

$$-p\left(\begin{array}{l}u_0(x,t)+\frac{4}{t}\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{2}{t^2}\frac{\partial U(x,t)}{\partial t}\\ -\frac{\partial^3 U(x,t)}{\partial t^3}\cdot\frac{\partial U(x,t)}{\partial x}-\frac{x}{t^2}-\frac{12}{t}\end{array}\right)=0, \tag{19}$$

Applying the inverse operator, $L^{-1}=\int_0^t\int_0^t\int_0^t(.)dtdtdt$, to both sides of Eq. (19), we get:

$$U(x,t)=U(x,0)+tU_t(x,0)+\frac{t^2}{2}U_{tt}(x,0)+\int_0^t\int_0^t\int_0^t u_0(x,t)dtdtdt$$

$$-p\left(\int_0^t\int_0^t\int_0^t\left[\begin{array}{l}u_0(x,t)+\frac{4}{t}\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{2}{t^2}\frac{\partial U(x,t)}{\partial t}\\ -\frac{\partial^3 U(x,t)}{\partial t^3}\cdot\frac{\partial U(x,t)}{\partial x}-\frac{x}{t^2}-\frac{12}{t}\end{array}\right]dtdtdt\right), \tag{20}$$

Suppose that the solution of Eq. (20), has the following form:

$$U(x,t)=U_0(x,t)+pU_1(x,t)+p^2U_2(x,t)+p^3U_3(x,t)+\cdots, \tag{21}$$

where $U_i(x,t)$, are functions which should be determined.

Substituting Eq. (21), into Eq. (20), and comparing coefficients of terms with identical powers of p, we obtain:

$$p^0 : U_0(x,t) = U(x,0) + tU_t(x,0) + \frac{1}{2}t^2 U_{tt}(x,0) + \int_0^t\int_0^t\int_0^t u_0(x,t)\,dtdtdt$$

$$p^1 : U_1(x,t) = \int_0^t\int_0^t\int_0^t \begin{pmatrix} -u_0(x,t) - \dfrac{4}{t}\dfrac{\partial^2 U_0(x,t)}{\partial t^2} - \dfrac{2}{t^2}\dfrac{\partial U_0(x,t)}{\partial t} \\ + \dfrac{\partial^3 U_0(x,t)}{\partial t^3}\cdot\dfrac{\partial U_0(x,t)}{\partial x} + \dfrac{x}{t^2} + \dfrac{12}{t} \end{pmatrix} dtdtdt$$

$$p^2 : U_2(x,t) = \int_0^t\int_0^t\int_0^t \begin{pmatrix} -\dfrac{4}{t}\dfrac{\partial^2 U_1(x,t)}{\partial t^2} - \dfrac{2}{t^2}\dfrac{\partial U_1(x,t)}{\partial t} \\ + \dfrac{\partial^3 U_1(x,t)}{\partial t^3}\cdot\dfrac{\partial U_1(x,t)}{\partial x} \end{pmatrix} dtdtdt, \tag{22}$$

$$\vdots$$

$$p^k : U_k(x,t) = \int_0^t\int_0^t\int_0^t \begin{pmatrix} -\dfrac{4}{t}\dfrac{\partial^2 U_{k-1}(x,t)}{\partial t^2} - \dfrac{2}{t^2}\dfrac{\partial U_{k-1}(x,t)}{\partial t} \\ + \dfrac{\partial^3 U_{k-1}(x,t)}{\partial t^3}\cdot\dfrac{\partial U_{k-1}(x,t)}{\partial x} \end{pmatrix} dtdtdt, \quad k = 3,4,\ldots$$

According to Eq. (15), we can give the first few Adomian's polynomials for the nonlinear term $\dfrac{\partial^3 U(x,t)}{\partial t^3}\cdot\dfrac{\partial U(x,t)}{\partial x}$ in Eq. (16), respectively,

$$\begin{aligned}
A_0 &= u_{0_{ttt}} u_{0_x} \\
A_1 &= u_{0_{ttt}} u_{1_x} + u_{1_{ttt}} u_{0_x} \\
A_2 &= u_{0_{ttt}} u_{2_x} + u_{1_{ttt}} u_{1_x} + u_{2_{ttt}} u_{0_x} \\
A_3 &= u_{0_{ttt}} u_{3_x} + u_{1_{ttt}} u_{2_x} + u_{2_{ttt}} u_{1_x} + u_{3_{ttt}} u_{0_x} \\
A_4 &= u_{0_{ttt}} u_{4_x} + u_{1_{ttt}} u_{3_x} + u_{2_{ttt}} u_{2_x} + u_{3_{ttt}} u_{1_x} + u_{4_{ttt}} u_{0_x} \\
&\vdots
\end{aligned} \tag{23}$$

To solve Eq. (22), for $U_0(x,t)$, and $U_1(x,t)$, we get:

$$U_0(x,t) = \frac{1}{2}xt + t^2 + \int_0^t\int_0^t\int_0^t u_0(x,t)\,dtdtdt$$

$$\begin{aligned}
U_1(x,t) = &\left(-\frac{1}{6}\alpha_0(x) - \frac{2}{3}\alpha_0(x) - \frac{1}{6}\alpha_0(x)\right)t^3 \\
&+ \left(-\frac{1}{24}\alpha_1(x) - \frac{1}{12}\alpha_1(x) + \frac{1}{48}\alpha_0(x) - \frac{1}{72}\alpha_1(x)\right)t^4 \\
&+ \left(-\frac{1}{60}\alpha_2(x) - \frac{1}{360}\alpha_2(x) - \frac{1}{45}\alpha_2(x) + \frac{1}{120}\alpha_1(x)\right)t^5 \\
&+ \begin{pmatrix} -\dfrac{1}{120}\alpha_3(x) - \dfrac{1}{120}\alpha_3(x) + \dfrac{1}{1200}\alpha_3(x) \\ + \dfrac{1}{720}\alpha_0(x)\alpha_2'(x) - \dfrac{1}{240}\alpha_2(x) \end{pmatrix} t^6 + \cdots = 0
\end{aligned}$$

Assuming that $u_0(x,t)=\sum_{n=0}^{\infty}\alpha_n(x)P_n(t),\ P_k(t)=t^k,\ k=0,1,2,\ldots,$

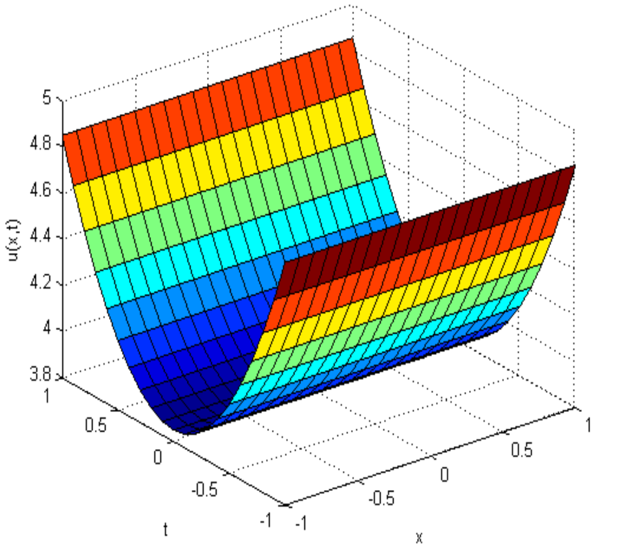
thus $U(x,0)=u(x,0), U_t(x,0)=u_t(x,0),$ and $U_{tt}(x,0)=u_{tt}(x,0).$ If we set the Taylor series of $U_1(x,t)=0,$ at $t=0,$ equal to zero then we have:

$$\alpha_0(x)=\alpha_1(x)=\alpha_2(x)=\alpha_3(x)=\alpha_4(x)=\alpha_5(x)=\alpha_6(x)=\cdots=0,$$

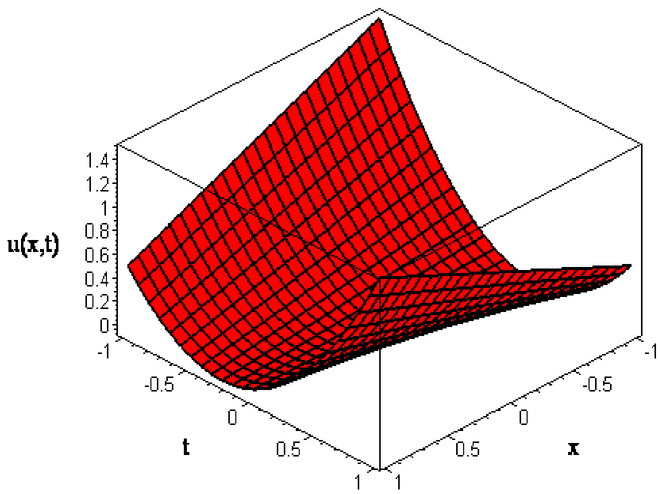
Therefore, the exact solution of Eq.(16), becomes as:

$$u(x,t)=\frac{xt}{2}+t^2$$

Fig.1:The surface shows the exact solution u(x,t) for example(3.1) using the method (NHPM)



Tig.1: Exact solution for example (3.1), using the method (NHPM)



Example (3.2):

We consider the following singular nonlinear initial value problem (IVP) of third order partial differential equations see [12],

$$\frac{\partial^3 u(x,t)}{\partial t^3}+\frac{9}{t}\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{18}{t^2}\frac{\partial u(x,t)}{\partial t}+\frac{6}{t^3}u(x,t)=\left(\frac{\partial u(x,t)}{\partial x}\right)^2+x+t, \qquad (24)$$

subject to the initial conditions,

$$u(x,0)=0, \qquad u_t(x,0)=0, \qquad u_{tt}(x,0)=0, \qquad (25)$$

To solve Eq. (24) by using (NHPM), we construct the following homotopy:

$$(1-p)\left(\frac{\partial^3 U(x,t)}{\partial t^3}-u_0(x,t)\right)+p\begin{pmatrix}\dfrac{\partial^3 U(x,t)}{\partial t^3}+\dfrac{9}{t}\dfrac{\partial^2 U(x,t)}{\partial t^2}+\dfrac{18}{t^2}\dfrac{\partial U(x,t)}{\partial t}\\ +\dfrac{6}{t^3}U(x,t)-\left(\dfrac{\partial U(x,t)}{\partial x}\right)^2-x-t\end{pmatrix}=0, \quad (26)$$

or

$$\frac{\partial^3 U(x,t)}{\partial t^3}=u_0(x,t)-p\begin{pmatrix}u_0(x,t)+\dfrac{9}{t}\dfrac{\partial^2 U(x,t)}{\partial t^2}+\dfrac{18}{t^2}\dfrac{\partial U(x,t)}{\partial t}\\ +\dfrac{6}{t^3}U(x,t)-\left(\dfrac{\partial U(x,t)}{\partial x}\right)^2-x-t\end{pmatrix}, \quad (27)$$

Applying the inverse operator, $L^{-1}=\int_0^t\int_0^t\int_0^t(.)dtdtdt,$ to both sides of Eq. (27), we have:

$$U(x,t)=U(x,0)+tU_t(x,0)+\frac{1}{2}t^2U_{tt}(x,0)+\int_0^t\int_0^t\int_0^t u_0(x,t)dtdtdt$$
$$-p\left(\int_0^t\int_0^t\int_0^t\begin{bmatrix}u_0(x,t)+\dfrac{9}{t}\dfrac{\partial^2 U(x,t)}{\partial t^2}+\dfrac{18}{t^2}\dfrac{\partial U(x,t)}{\partial t}\\ +\dfrac{6}{t^3}U(x,t)-\left(\dfrac{\partial U(x,t)}{\partial x}\right)^2-x-t\end{bmatrix}dtdtdt\right), \quad (28)$$

Suppose that the solution of Eq. (24), has the following form:

$$U(x,t)=U_0(x,t)+pU_1(x,t)+p^2U_2(x,t)+p^3U_3(x,t)+\cdots, \quad (29)$$

where $U_i(x,t),$ are functions which should be determined.

Substituting Eq. (29), into Eq. (28), and comparing coefficients of terms with identical powers of $p,$ we obtain:

$$p^0:U_0(x,t)=U(x,0)+tU_t(x,0)+\frac{1}{2!}t^2U_{tt}(x,0)+\int_0^t\int_0^t\int_0^t u_0(x,t)dtdtdt$$

$$p^1:U_1(x,t)=\int_0^t\int_0^t\int_0^t\begin{bmatrix}-u_0(x,t)-\dfrac{9}{t}\dfrac{\partial^2 U(x,t)}{\partial t^2}-\dfrac{18}{t^2}\dfrac{\partial U(x,t)}{\partial t}\\ -\dfrac{6}{t^3}U(x,t)+\left(\dfrac{\partial U(x,t)}{\partial x}\right)^2+x+t\end{bmatrix}dtdtdt$$

$$p^2 : U_2(x,t) = \int_0^t\int_0^t\int_0^t \left[\begin{array}{l} -\frac{9}{t}\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{18}{t^2}\frac{\partial U(x,t)}{\partial t} \\ -\frac{6}{t^3}U(x,t) + \left(\frac{\partial U(x,t)}{\partial x}\right)^2 \end{array} \right] dtdtdt, \tag{30}$$

$$\vdots$$

$$p^k : U_k(x,t) = \int_0^t\int_0^t\int_0^t \left[\begin{array}{l} -\frac{9}{t}\frac{\partial^2 U_{k-1}(x,t)}{\partial t^2} - \frac{18}{t^2}\frac{\partial U_{k-1}(x,t)}{\partial t} \\ -\frac{6}{t^3}U_{k-1}(x,t) + \left(\frac{\partial U_{k-1}(x,t)}{\partial x}\right)^2 \end{array} \right] dtdtdt,$$

According to Eq. (15), we can gives the first few Adomian's polynomials for the nonlinear term $\left(\frac{\partial U(x,t)}{\partial x}\right)^2$ in Eq. (24), respectively,

$$\begin{aligned}
A_0 &= u_{0_x}^2 \\
A_1 &= 2u_{0_x}u_{1_x} \\
A_2 &= u_{1_x}^2 + 2u_{0_x}u_{2_x} \\
A_3 &= 2u_{0_x}u_{3_x} + 2u_{1_x}u_{2_x} \\
A_4 &= 2u_{0_x}u_{4_x} + 2u_{1_x}u_{3_x} + u_{2_x}^2 \\
A_5 &= 2u_{0_x}u_{5_x} + 2u_{1_x}u_{4_x} + 2u_{2_x}u_{3_x} \\
&\vdots
\end{aligned} \tag{31}$$

To solve Eq. (30), for $U_0(x,t)$, and $U_1(x,t)$, we get:

$$U_0(x,t) = \int_0^t\int_0^t\int_0^t u_0(x,t)\,dtdtdt$$

$$\begin{aligned}
U_1(x,t) = &\left(-\frac{1}{6}\alpha_0(x) - \frac{3}{2}\alpha_0(x) - \frac{3}{2}\alpha_0(x) - \frac{1}{6}\alpha_0(x) + \frac{1}{6}x\right)t^3 \\
&+ \left(-\frac{1}{24}\alpha_1(x) - \frac{3}{16}\alpha_1(x) - \frac{1}{8}\alpha_1(x) - \frac{1}{96}\alpha_1(x) + \frac{1}{24}\right)t^4 \\
&+ \left(-\frac{1}{60}\alpha_2(x) - \frac{1}{20}\alpha_2(x) - \frac{1}{40}\alpha_2(x) - \frac{1}{600}\alpha_2(x)\right)t^5 \\
&+ \left(-\frac{1}{120}\alpha_3(x) - \frac{3}{160}\alpha_3(x) - \frac{3}{400}\alpha_3(x) - \frac{1}{2400}\alpha_3(x)\right)t^6 \\
&+ \left(-\frac{1}{210}\alpha_4(x) - \frac{3}{350}\alpha_4(x) - \frac{1}{350}\alpha_4(x) - \frac{1}{7350}\alpha_4(x)\right)t^7 \\
&+ \left(-\frac{1}{336}\alpha_5(x) - \frac{1}{224}\alpha_5(x) - \frac{1}{784}\alpha_5(x) - \frac{1}{18816}\alpha_5(x)\right)t^8
\end{aligned}$$

$$+\begin{pmatrix} -\frac{1}{504}\alpha_6(x)-\frac{1}{392}\alpha_6(x)-\frac{1}{1568}\alpha_6(x) \\ -\frac{1}{42336}\alpha_6(x)+\frac{1}{51840}(\alpha_0')^2 \end{pmatrix} t^9+\cdots=0$$

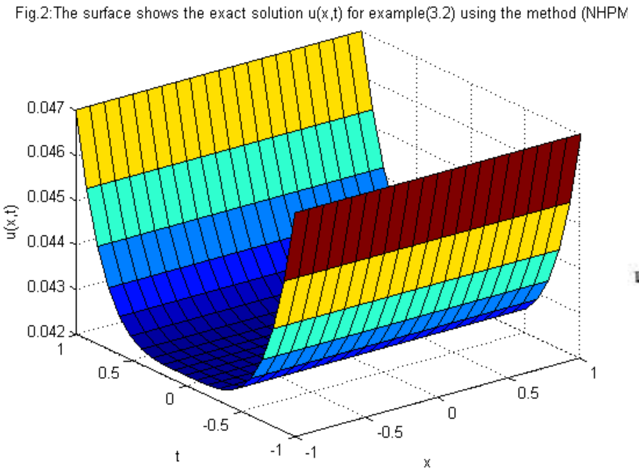
Assuming that $u_0(x,t)=\sum_{n=0}^{\infty}\alpha_n(x)P_n(t),\ P_k(t)=t^k,\ k=0,1,2,\ldots,$

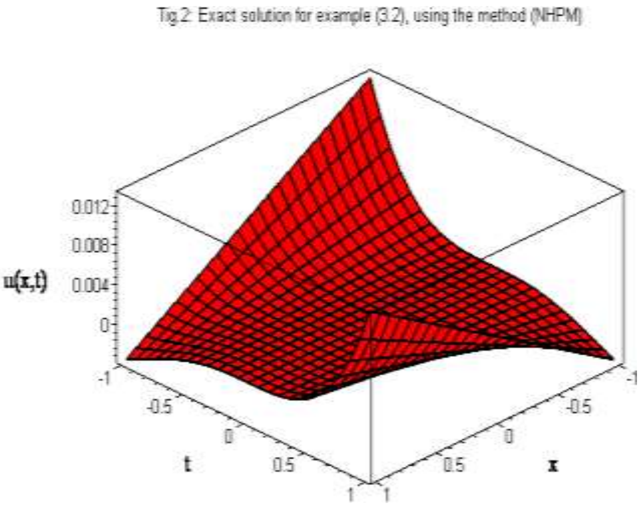
thus $U(x,0)=u(x,0), U_t(x,0)=u_t(x,0),$ and $U_{tt}(x,0)=u_{tt}(x,0).$ If we set the Taylor series of $U_1(x,t)=0,$ at $t=0,$ equal to zero then we have

$$\alpha_0(x)=\frac{x}{20},\alpha_1(x)=\frac{4}{35},\alpha_2(x)=\alpha_3(x)=\alpha_4(x)=\alpha_5(x)=0,$$
$$\alpha_6(x)=\frac{7}{264000},\alpha_7(x)=\alpha_8(x)=\alpha_9(x)=\cdots=0.$$

Therefore, the exact solution of Eq.(24), becomes as:

$$u(x,t)=\frac{xt^3}{120}+\frac{t^4}{210}+\frac{t^9}{19008000}$$





Example (3.3):

Consider the Blasius equations see [6],

$$u'''(x)+\frac{1}{2}u(x)u''(x)=0, \tag{32}$$

subject to the initial conditions,

$$u(0)=0, \qquad u'(0)=1, \qquad u'\to 0, x\to\infty \tag{33}$$

To solve Eq. (32), we consider an extra initial condition, that is, $u''(0)=\beta.$ In order to solve Eq. (32), with this extra initial condition, using method (NHPM), we construct the following homotopy:

$$(1-p)\left(U'''(x)-u_0(x)\right)+p\left(U'''(x)+\frac{1}{2}U(x)U''(x)\right)=0, \tag{34}$$

or

$$U'''(x)=u_0(x)-p\left(u_0(x)+\frac{1}{2}U(x)U''(x)\right), \tag{35}$$

Applying the inverse operator, $L^{-1}=\int_0^x\int_0^x\int_0^x(.)dxdxdx$, to both sides of Eq. (35), we have:

$$U(x)=U(0)+x\,U'(0)+\frac{x^2}{2!}U''(0)+\int_0^x\int_0^x\int_0^x u_0(x)dxdxdx$$
$$-p\left(\int_0^x\int_0^x\int_0^x\left[u_0(x)+\frac{1}{2}U(x)U''(x)\right]dxdxdx\right), \tag{36}$$

Suppose that the solution of Eq. (35), has the following form

$$U(x)=U_0(x)+pU_1(x)+p^2U_2(x)+p^3U_3(x)+\cdots, \tag{37}$$

where $U_i(x)$, are functions which should be determined.

Substituting Eq. (37), into Eq. (36), and comparing coefficients of terms with identical powers of p, we obtain:

$$p^0: U_0(x)=U(0)+x\,U'(0)+\frac{x^2}{2!}U''(0)+\int_0^x\int_0^x\int_0^x u_0(x)dxdxdx$$
$$p^1: U_1(x)=\int_0^x\int_0^x\int_0^x\left[u_0(x)+\frac{1}{2}U_0(x)U_0''(x)\right]dxdxdx$$
$$p^2: U_2(x)=\int_0^x\int_0^x\int_0^x\left[\frac{1}{2}U_0(x)U_2''(x)+\frac{1}{2}U_1(x)U_1''(x)\right]dxdxdx, \tag{38}$$
$$\vdots$$

According to Eq. (15), we can gives the first few Adomian's polynomials for the nonlinear term $\frac{1}{2}U(x)U''(x)$, in Eq. (32), respectively,

$$A_0=\frac{1}{2}u_0u_{0_{xx}}$$
$$A_1=\frac{1}{2}u_0u_{1_{xx}}+\frac{1}{2}u_1u_{0_{xx}}$$
$$A_2=\frac{1}{2}u_0u_{2_{xx}}+\frac{1}{2}u_1u_{1_{xx}}+\frac{1}{2}u_2u_{0_{xx}}$$
$$A_3=\frac{1}{2}u_0u_{3_{xx}}+\frac{1}{2}u_1u_{2_{xx}}+\frac{1}{2}u_2u_{1_{xx}}+\frac{1}{2}u_3u_{0_{xx}}$$
$$A_4=\frac{1}{2}u_0u_{4_{xx}}+\frac{1}{2}u_1u_{3_{xx}}+\frac{1}{2}u_2u_{2_{xx}}+\frac{1}{2}u_3u_{1_{xx}}+\frac{1}{2}u_4u_{0_{xx}}$$
$$A_5=\frac{1}{2}u_0u_{5_{xx}}+\frac{1}{2}u_1u_{4_{xx}}+\frac{1}{2}u_2u_{3_{xx}}+\frac{1}{2}u_3u_{2_{xx}}+\frac{1}{2}u_4u_{1_{xx}}+\frac{1}{2}u_5u_{0_{xx}} \tag{39}$$
$$\vdots$$

To solve Eq. (38), for $U_0(x)$, and $U_1(x)$, we get:

$$U_0(x) = x + \frac{\beta x^2}{2} + \int_0^x\int_0^x\int_0^x u_0(x)\,dxdxdx$$

$$U_1(x) = \left(-\frac{1}{6}\alpha_0\right)x^3 + \left(-\frac{1}{24}\alpha_1 - \frac{1}{48}\beta\right)x^4 + \left(-\frac{1}{60}\alpha_2 - \frac{1}{120}\alpha_0 - \frac{1}{240}\beta^2\right)x^5$$

$$+\left(-\frac{1}{120}\alpha_3 - \frac{1}{480}\alpha_1 - \frac{1}{1440}\alpha_0\beta - \frac{1}{480}\alpha_0\beta\right)x^6$$
$$+\left(-\frac{1}{210}\alpha_4 - \frac{1}{1260}\alpha_2 - \frac{1}{10080}\alpha_1\beta - \frac{1}{1680}\alpha_1\beta - \frac{1}{2520}\alpha_0^2\right)x^7$$
$$+\left(-\frac{1}{336}\alpha_5 - \frac{1}{2688}\alpha_3 - \frac{1}{40320}\alpha_2\beta - \frac{1}{4032}\alpha_1\beta - \frac{1}{16128}\alpha_0\alpha_1 - \frac{1}{8064}\alpha_0\alpha_1\right)x^8 + \cdots = 0$$

Assuming that $u_0(x) = \sum_{n=0}^{\infty}\alpha_n P_n(x),\ P_k(t) = t^k,\ k = 0,1,2,\ldots,$

thus $U(x) = u(x), U_t(x) = u_t(x),$ and $U_{tt}(x) = u_{tt}(x).$ If we set the Taylor series of $U_1(x) = 0,$ at $x = 0,$ equal to zero then we have:

$$\alpha_0(x) = 0, \alpha_1(x) = -\frac{\beta}{2}, \alpha_2(x) = -\frac{\beta^2}{4}, \alpha_3(x) = \frac{\beta}{8}, \alpha_4(x) = \frac{11\beta^2}{96}, \ldots$$

Therefore, the approximate series solution of Eq.(14), becomes as:

$$u(x) = x + \frac{\beta x^2}{2} - \frac{\beta x^4}{48} - \frac{\beta^2 x^5}{240} + \frac{\beta x^6}{960} + \frac{11\beta^2 x^7}{20160} + \frac{11\beta^3 x^8}{161280} - \cdots$$

This is exactly the same as that obtained by Abdul-Majid Wazwaz (2007) through (VIM).

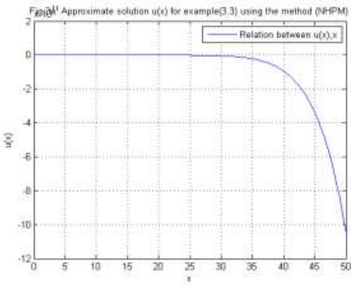


Fig.3 Approximate solution u(x) for example(3.3) using the method (NHPM)

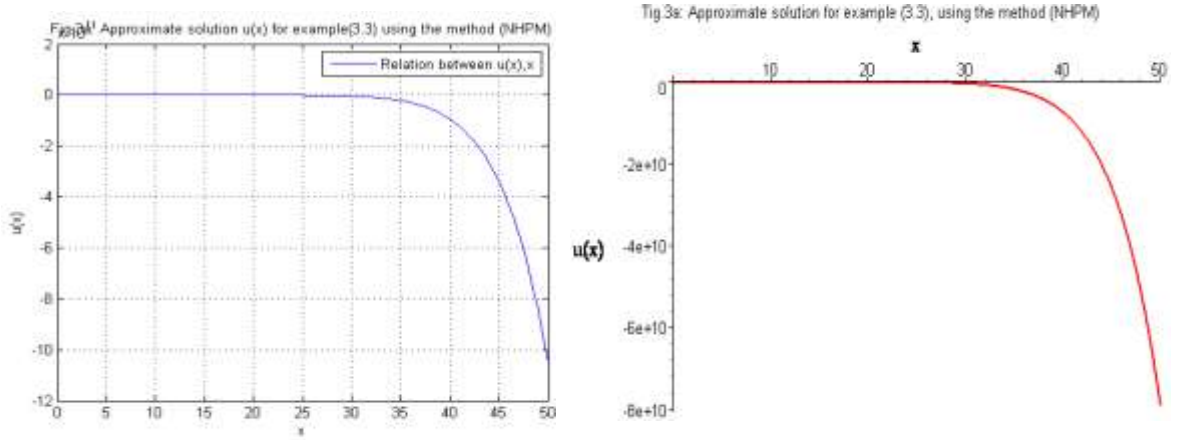


Figure 3a: Approximate solution of example (3.3) for $\beta = 1$

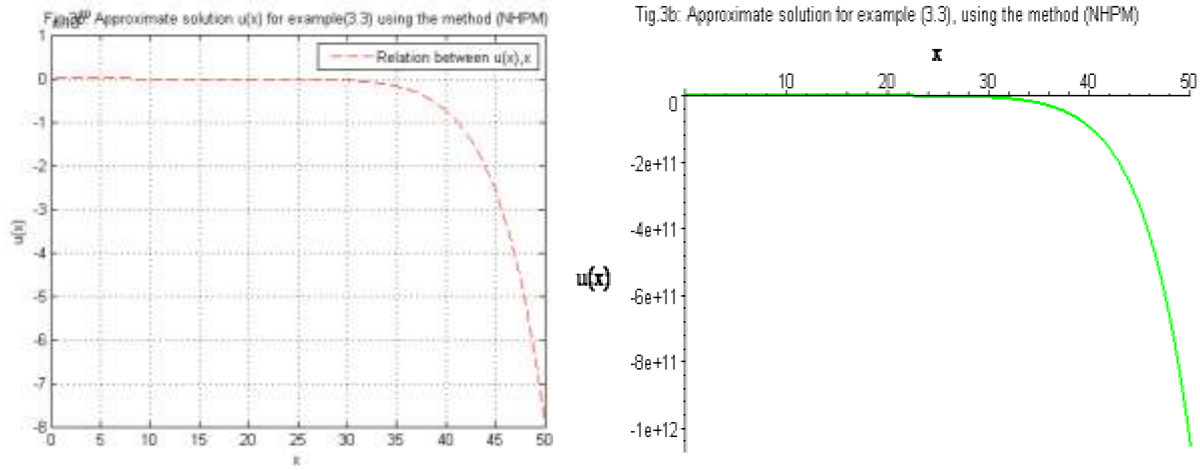


Figure 3b: Approximate solution of example (3.3) for $\beta = 0.5$

Example (3.4):

Consider the non-homogeneous equation see [6],

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + \eta(x,t), \tag{40}$$

where $\eta(x,t) = 2\pi^2 e^{-\pi t} \sin(\pi x)$, subject to the initial conditions,

$$u(x,0) = \sin(\pi x), \qquad u_t(x,0) = -\pi \sin(\pi x), \tag{41}$$

To solve Eq. (40) by using (NHPM), we construct the following homotopy:

$$(1-p)\left(\frac{\partial^2 U(x,t)}{\partial t^2} - u_0(x)\right) + p\left(\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{\partial^2 U(x,t)}{\partial x^2} - 2\pi^2 e^{-\pi t} \sin(\pi x)\right) = 0, \tag{42}$$

or

$$\frac{\partial^2 U(x,t)}{\partial t^2} = u_0(x) - p\left(u_0(x) - \frac{\partial^2 U(x,t)}{\partial x^2} - 2\pi^2 e^{-\pi t}\sin(\pi x)\right) = 0, \qquad (43)$$

Applying the inverse operator, $L^{-1} = \int_0^t\int_0^t (.)dtdt$, to both sides of Eq. (43), we have:

$$U(x,t) = U(x,0) + tU'(x,0) + \int_0^t\int_0^t u(x,t)dtdt$$
$$- p\int_0^t\int_0^t\left(u(x,t) - \frac{\partial^2 U(x,t)}{\partial x^2} - 2\pi^2 e^{-\pi t}\sin(\pi x)\right)dtdt, \qquad (44)$$

Suppose that the solution of Eq. (40), has the following form,

$$U(x,t) = U_0(x,t) + pU_1(x,t) + p^2 U_2(x,t) + p^3 U_3(x,t) + \cdots, \qquad (45)$$

where $U_i(x,t)$, are functions which should be determined.

Substituting Eq. (45), into Eq. (44), and comparing coefficients of terms with identical powers of p, we obtain:

$$p^0 : U_0(x,t) = U(x,0) + tU'(x,0) + \int_0^t\int_0^t u(x,t)dtdt,$$
$$p^1 : U_1(x,t) = \int_0^t\int_0^t\left(-u(x,t) + \frac{\partial^2 U_0(x,t)}{\partial x^2} + 2\pi^2 e^{-\pi t}\sin(\pi x)\right)dtdt$$
$$p^2 : U_2(x,t) = \int_0^t\int_0^t\left(\frac{\partial^2 U_1(x,t)}{\partial x^2}\right)dtdt$$
$$\vdots$$
$$p^k : U_k(x,t) = \int_0^t\int_0^t\left(\frac{\partial^2 U_{k-1}(x,t)}{\partial x^2}\right)dtdt, \qquad (46)$$

To solve Eq. (46), for $U_0(x,t)$, and $U_1(x,t)$, we get

$$U_0(x,t) = \sin(\pi x) - \pi t \sin(\pi x) + \int_0^t \int_0^t u(x,t)dtdt,$$

$$U_1(x,t) = \left(-\frac{1}{2}\alpha_0(x) - \frac{1}{2}\pi^2 \sin(\pi x)\right)t^2$$
$$+\left(-\frac{1}{6}\alpha_1(x) + \frac{1}{6}\pi^3 \sin(\pi x)\right)t^3$$
$$+\left(-\frac{1}{12}\alpha_2(x) + \frac{1}{24}\alpha_0''(x)\right)t^4$$
$$+\left(-\frac{1}{20}\alpha_3(x) + \frac{1}{120}\alpha_1''(x)\right)t^5$$
$$+\left(-\frac{1}{30}\alpha_4(x) + \frac{1}{360}\alpha_2''(x)\right)t^6 + \cdots = 0$$

Assuming that $u_0(x,t) = \sum_{n=0}^{\infty} \alpha_n(x)P_n(t),\ P_k(t) = t^k,\ k = 0,1,2,\ldots,$

thus $U(x,0) = u(x,0), U_t(x,0) = u_t(x,0),$ and $U_{tt}(x,0) = u_{tt}(x,0).$ If we set the Taylor series of $U_1(x,t) = 0,$ at $t = 0,$ equal to zero then we have

$$\alpha_0(x) = \pi^2 \sin(\pi x),\quad \alpha_1(x) = -\pi^3 \sin(\pi x),\ \alpha_2(x) = \frac{1}{2}\pi^4 \sin(\pi x),$$
$$\alpha_3(x) = -\frac{1}{6}\pi^5 \sin(\pi x),\ \alpha_4(x) = -\frac{1}{6}\pi^5 \sin(\pi x), \ldots$$

this implies that

$$u(x) = \sin(\pi x)\left(1 - \pi t + \frac{\pi^2 t^2}{2!} - \frac{\pi^3 t^3}{3!} + \frac{\pi^4 t^4}{4!} - \frac{\pi^5 t^5}{5!} + \frac{\pi^6 t^6}{6!} - \frac{\pi^7 t^7}{7!} + \cdots\right)$$

Therefore, the exact solution of Eq.(40), is

$$u(x,t) = e^{-\pi t} \sin(\pi x),$$

(4) Conclusions:

In this article, systems of third order nonlinear singular partial differential equations are solved by using a new homotopy perturbation method (NHPM). This method has been applied to four examples successfully, numerical results reveal that method is powerful tool and gives quickly convergent approximations solution that to exact solution for nonlinear partial differential equations.

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