

Boundary Gradient Detectability in Parameter Systems

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Abstract

The principal reason behind introducing this notion is that the possibility to design a dynamic system (may be called regional boundary gradient observer) which enable to estimate the unknown system state gradient. Then for linear infinite dimensional systems in a Hilbert space, we give various new results related with different measurements. In addition, we provided a description of the regional boundary gradient strategic sensors for completion the regional boundary gradient observability and regional boundary gradient detectability. Finally, we present and illustrate some applications of sensors structures which relate by regional boundary gradient detectability in diffusion distributed parameter systems.

Keywords: Γ_G -strategic sensors, Γ_G -observer, Γ_G -detectability, infinite dimensional systems, Hilbert space.

1. Introduction

Many real problems in the sensors and detectability of distributed parameter systems can be reformulated as problems of infinite dimensional systems in a domain Ω [1-3]. The analysis of distributed parameter systems consists a set of notions as observability, detectability, stability, and observer is represented by infinite dimensional systems of partial differential equations [4-8]. Recently, the concept of regional and regional boundary analysis has been introduced by El Jai, Zerrik, Al-Saphory *et al.*, for finite time horizon [9-13] and for infinite time horizon [14-16]. Thus, this concept which is given important tool to solve many problems in real world not in total domain of the considered system state but in sub-region of the domain or of the boundary [17-19]. Later, another direction of regional analysis has been extended is the system state gradient for observability, detectability and strategic sensors [20-26].

The purpose of this paper is to study and investigate the concept of regional boundary gradient detectability by using the choice of sensors. The principle reason for considering this case is that, firstly exist systems which are detectable on some boundary sub-region but are not detectable in any neighborhood of $\Gamma \subset \partial\Omega$. Secondly, it is closer to a real situation, the treatment of water by using a bioreactor where the objective is to detect the concentration of substrate at the boundary output of the bioreactor in order the water regulation is achieved (for example see figure 1)[27].

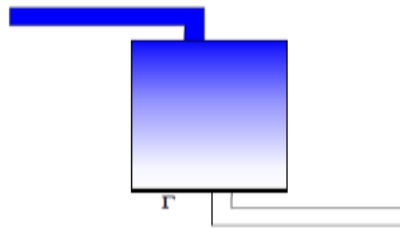


Fig. 1: Detection of substrate concentration at the boundary output of the reactor.

The outline of this paper is organized as follow.

Section 2 concerns the class of considered system, definition, characterizations in connection with sensors and preliminaries of regional boundary gradient observability and detectability. Section 3, devotes to the problem of crossing method from internal region to boundary case by using trace operator estimate the gradient of state in region Γ . Section 4, gives an applications to various situations of sensors locations on the regional boundary gradient detectability in diffusion parabolic distributed systems. Finally, section tackles the relation between regional boundary detectability of state gradient and regional boundary observer.

2. Regional Boundary Gradient Detectability

In this section, we extend the results in [15, 21] to the boundary regional case by considering $\Gamma \subset \partial\Omega$. Thus, we will give some definitions which will be used to explain the notion of regional boundary gradient observability and detectability in $(H^{1/2}(\Gamma))^n$ (statespace).

2.1 Considered system and preliminary

Let Ω be an open bounded subset of R^n with smooth boundary $\partial\Omega$ and Γ be a sub-boundary of $\partial\Omega$. We considered the parabolic system is described by the following state space equations

$$\begin{cases} \frac{\partial z}{\partial t}(\mu, t) = Az(\mu, t) + Bu(t) \Omega \times]0, T[\\ z(\mu, 0) = z_0(\mu) \bar{\Omega} \\ \frac{\partial z}{\partial \nu}(\eta, t) = 0 \end{cases} \quad \partial\Omega \times]0, T[\quad (1)$$

where $\mu \in \Omega$, $\eta \in \partial\Omega$, $t \in [0, T]$ and $(\mu, t) \in \Omega \times]0, T[$, $(\eta, t) \in \partial\Omega \times]0, T[$, $(\mu, 0) \in \bar{\Omega}$,

Augmented with the output function

$$y(\cdot, t) = Cz(\cdot, t) \quad (2)$$

- The Hilbert spaces Z, U and \mathcal{O} are separable where $Z = H^1(\bar{\Omega})$ is the state space, $\bar{\Omega}$ is the closure of Ω , $U = L^2(0, T, R^p)$ is the control space and $\mathcal{O} = L^2(0, T, R^q)$ is the observation space, where p and q are the numbers of actuators and sensors.

• $A = \sum_{i,j=1}^n \frac{\partial}{\partial z_j} \left(a_{ij} \frac{\partial}{\partial z_j} \right)$ with $a_{ij} \in D(\bar{A})$ (domain of \bar{A}) is a second order linear differential operator, which generates a strongly continuous semi-group $(S_A(t))_{t \geq 0}$ on the state space Z and is self-adjoint with compact resolvent.

• The operators $B \in L(R^p, Z)$ and $C \in L(Z, R^q)$ are depend on the structure of actuators and sensors as in [8] see (figure 2) which is a mathematical model is more general spatial case in (figure 1).

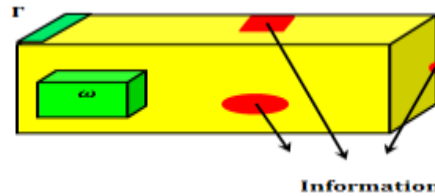


Fig. 2: The domain Ω , the boundary regions Γ , and sensors locations.

• Under the given assumption above, the system (1) has a unique solution given by the following form [1-2].

$$z(\mu, t) = S_A(t)z_0(\mu) + \int_0^t S_A(t-s)Bu(s) ds \quad (3)$$

• The problem is how to detect the current state in a given sub-boundary Γ , and to give a sufficient condition for the existence of a regional boundary gradient detectability.

• The measurements can be obtained by the use of zone or pointwise sensors, which may be located in Ω or $\partial\Omega$ [3].

• We first recall a sensors are defined by any couple $(D_i, f_i)_{1 \leq i \leq q}$ where D_i be a non-empty closed subsets of $\bar{\Omega}$, which is spatial supports of sensors and $f_i \in L^2(D_i)$ represent the distributions of the sensing measurements on D_i . Then, according to the choice of the parameters D_i and f_i , we have different types of sensors:

• It may be zone, if $D_i \subset \bar{\Omega}$ and $f_i \in L^2(D_i)$. In this case, the operator C is bounded and the output function (2) may be given by the form

$$y(t) = \int_{D_i} z(\mu, t) f_i(\mu) d\mu \quad (4)$$

• It may be pointwise, if $D_i = \{b_i\}$ with $b_i \in \bar{\Omega}$ and $f = \delta(\cdot - b_i)$, where δ is the Dirac mass concentrated in b . In this case, the operator C is un bounded and the output function (2) may be given by the form

$$y(t) = \int_{\Omega} z(\mu, t) \delta_{b_i}(\mu - b_i) d\mu \quad (5)$$

• It may be boundary zone, if $\Gamma_i \subset \partial\Omega$ and $f_i \in L^2(\Gamma_i)$, the output function (2) may be given by the form

$$y(t) = \int_{\Gamma_i} z(\eta, t) f_i(\eta) d\eta \quad (6)$$

• The initial state z_0 and its gradient ∇z_0 are supposed to be unknown, the problem concerns the reconstruction of the initial gradient ∇z_0 on the sub-region Γ of the system domain $\partial\Omega$.

• Now, we consider the operator K given by the form

$$K: Z \rightarrow \mathcal{O}$$

$$z \rightarrow CS_A(\cdot) z$$

where K is bounded linear operator as in [7-8]. Thus, the adjoint operator K^* of K is defined by

$K^*: \mathcal{O} \rightarrow Z$, and represented by the form

$$K^* y^* = \int_0^t S_A^*(s) C^* y^*(s) ds$$

• The operator ∇ denotes the gradient is given by

$$\begin{cases} \nabla: H^1(\Omega) \rightarrow (H^1(\Omega))^n \\ z \rightarrow \nabla z = \left(\frac{\partial z}{\partial \mu_1}, \dots, \frac{\partial z}{\partial \mu_n} \right) \end{cases}$$

with the adjoint of ∇ denotes by ∇^* is given by

$$\begin{cases} \nabla^*: (H^1(\Omega))^n \rightarrow H^1(\Omega) \\ z \rightarrow \nabla_z^* = v \end{cases}$$

where v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = -\text{div}(z) \text{ in } \Omega \\ v = 0 \quad \text{in } \partial\Omega \end{cases}$$

• The trace operator of order zero is described by [28]

$$\gamma_0: H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

which is linear, surjective and continuous [2]. Thus, the extension of the trace operator of order zero which is denoted by γ defined as

$$\gamma: (H^1(\Omega))^n \rightarrow (H^{1/2}(\partial\Omega))^n$$

and the adjoints are respectively given by γ_0^*, γ^* .

• For a sub-boundary Γ of $\partial\Omega$ and let $\tilde{\chi}_\Gamma$ be the function defined by

$$\begin{cases} \tilde{\chi}_\Gamma: H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\Gamma) \\ z \rightarrow \chi_\Gamma z = z|_\Gamma \end{cases}$$

With $z|_\Gamma$ is the restriction of the state z to Γ , and

$$\chi_\Gamma: (H^{1/2}(\partial\Omega))^n \rightarrow (H^{1/2}(\Gamma))^n$$

where the adjoints are respectively given by $\tilde{\chi}_\Gamma^*, \chi_\Gamma^*$.

• Finally, we introduced the operator $H_{\Gamma_G} = \chi_\Gamma \gamma \nabla K^*$ from \mathcal{O} into $(H^{1/2}(\Gamma))^n$ and the adjoint of this operator given by $H_{\Gamma_G}^* = K \nabla^* \gamma^* \chi_\Gamma^*$.

2.2 Definitions and characterizations

In this sub-section, we introduce some definitions and descriptions of regional boundary gradient observability, detectability and strategic sensors, which is derived of [20-26]. Consider the autonomous system of (1) define by

$$\begin{cases} \frac{\partial z}{\partial t}(\mu, t) = Az(\mu, t) \Omega \times]0, T[\\ z(\mu, 0) = z_0(\mu) \bar{\Omega} \\ \frac{\partial z}{\partial v}(\eta, t) = 0 \quad \partial\Omega \times]0, T[\end{cases} \quad (7)$$

The solution of (7) is given by the following form

$$z(\mu, t) = S_A(t)z_0(\mu) \quad \text{for all } t \in [0, T] \quad (8)$$

Definition 2.1:(a) The system (7) augmented with the output function (2) (or the systems (7)-(2)) are said to be an exactly regionally boundary gradient observable on Γ , if

$$\text{Im } H_{\Gamma_G} = (H^{1/2}(\Gamma))^n$$

(b) The systems (7)-(2) are said to be an approximately regionally boundary gradient observable on Γ , if

$$\overline{\text{Im } H_{\Gamma_G}} = (H^{1/2}(\Gamma))^n$$

Now, we give a notion of the regional boundary gradient strategic sensors.

Definition 2.2:A sensor (D, f) is said to be regionally boundary gradient strategic on Γ (or Γ_G -strategic), if the observed systems are an approximately Γ_G -observable.

Definition 2.3: The semi-group $(S_A(t))_{t \geq 0}$ is said to be regionally boundary gradient stable (or Γ_G -stable) on the space $(H^{1/2}(\Gamma))^n$, if for some positive constants M_{Γ_G} and ω_{Γ_G} , then

$$\|\chi_\Gamma \gamma \nabla S_A(\cdot)\|_{L((H^{1/2}(\Gamma))^n, H^1(\bar{\Omega}))} \leq M_{\Gamma_G} e^{-\omega_{\Gamma_G} t}, t \geq 0 \quad (9)$$

Remark 2.4: If the semi-group $(S_A(t))_{t \geq 0}$ is Γ_G -stable on $(H^{1/2}(\Gamma))^n$, then for all $z_0(\cdot) \in H^1(\bar{\Omega})$, the solution of the system (7) satisfies

$$\lim_{t \rightarrow \infty} \|\chi_{\Gamma} \nabla S_A(\cdot, t)\|_{(H^{1/2}(\Gamma))^n} = \lim_{t \rightarrow \infty} \|\chi_{\Gamma} \nabla S_A(t) z_0\|_{(H^{1/2}(\Gamma))^n} = 0 \quad (10)$$

Definition 2.5: The system (7) is Γ_G -stable, if the operator A generates a semi-group which is Γ_G -stable on $(H^{1/2}(\Gamma))^n$.

Definition 2.6: The systems (1)-(2) are said to be regionally boundary gradient detectable on Γ (or Γ_G -detectable), if there exists an operator $H_{\Gamma_G}: R^q \rightarrow (H^{1/2}(\Gamma))^n$, such that the operator $(A - H_{\Gamma_G} C)$ generates a strongly continuous semi-group $(S_{H_{\Gamma_G}}(t))_{t \geq 0}$, which is Γ_G -stable on $(H^{1/2}(\Gamma))^n$.

Proposition 2.7: If the systems (1)-(2) are an exactly Γ_G -observable, then it is Γ_G -detectable. This results gives the following inequality: $\exists k > 0$, such that

$$\|\chi_{\Gamma} \nabla S_A(\cdot) z\|_{(H^{1/2}(\Gamma))^n} \leq k \|CS_A(\cdot) z\|_{L^2(0, \infty, \rho)}, \text{ for all } z \in (H^{1/2}(\Gamma))^n \quad (11)$$

Proof: We conclude the proof of this proposition is conclude from the results on observability considering $\chi_{\Gamma} \nabla K^*$. We have the following forms [2]

1- $Imf \subset Img$.

2- There exists $k > 0$, such that

$$\|f^* z^*\|_{E^*} \leq k \|g^* z^*\|_{F^*}, \text{ for all } z^* \in G^*$$

From the right hand said of above inequality $k \|g^* z^*\|_{F^*}$, there exists $M_{\Gamma_G}, \omega_{\Gamma_G} > 0$ with $k < M_{\Gamma_G}$, such that

$$k \|g^* z^*\|_{F^*} \leq M_{\Gamma_G} e^{-\omega_{\Gamma_G} t} \|z^*\|_{F^*}$$

where E, F and G be a reflexive Banach spaces and $f \in L(E, G)$, $g \in L(F, G)$. If we apply this result, considered

$$E = G = (H^{1/2}(\Gamma))^n, \quad F = \mathcal{O}, \quad f = Id_{(H^{1/2}(\Gamma))^n}$$

and

$$g = S_A^*(\cdot) \chi_{\Gamma}^* \gamma^* \nabla^* C^*$$

where $S_A(\cdot)$ is a strongly continuous semi-group generates by A , which is Γ_G -stable, then it is Γ_G -detectable. Thus, the notion of Γ_G -detectability is a weaker property than the exact Γ_G -observability [14-15].

2.3 Γ_G -Strategic sensors and Γ_G -detectability

In the sub-section, we shall develop the characterization result that links an Γ_G -detectable and sensors structures. For that purpose, we assume that the operator A has a complete set of eigenfunctions $H^1(\bar{\Omega})$ (which is Sobolev space of order one) [1] denoted φ_{m_j} orthonormal in $(H^{1/2}(\Gamma))^n$ and the associated eigenvalues λ_m are of multiplicity r_m and suppose that the system (1) has J unstable modes.

Thus, the sufficient condition of an Γ_G -detectability is given by the following theorem.

Theorem 2.8: Suppose that there are q zone sensors $(D_i, f_i)_{1 \leq i \leq q}$ and the spectrum of A contains J eigenvalues with non-negative real parts. The systems (1)-(2) are Γ_G -detectable if and only if

1- $q \geq m$,

2- $\text{rank } G_i = m_i$, for all $i, i = 1, \dots, J$ with

$$G = (G)_{ij} = \begin{cases} \langle \psi_j(\cdot), f_i(\cdot) \rangle_{L^2(D_i)} & \text{zone sensors} \\ \psi_j(b_i) & \text{pointwise sensors} \\ \langle \frac{\partial \psi_j}{\partial \nu}, f_i(\cdot) \rangle_{L^2(\Gamma_i)} & \text{boundary zone sensors} \end{cases}$$

where $\sup m_i = m < \infty$ and $j = 1, \dots, m_i$.

Proof: For brevity, the proof is limited to the case of zone sensors. Under the assumptions of section 2.1, the system (1) may be decomposed by the projections P and $I - P$ on two parts, unstable and stable [7]. The state

vector may be given by $z(\mu, t) = [z_1(\mu, t)z_2(\mu, t)]^{tr}$, where $z_1(\mu, t)$ is the state component of the unstable part of the system (1) that may be written in the form

$$\begin{cases} \frac{\partial z_1}{\partial t}(\mu, t) = A_1 z_1(\mu, t) + PBu(t) & \Omega \times]0, T[\\ z_1(\mu, 0) = z_{1_0}(\mu)\bar{\Omega} \\ \frac{\partial z_1}{\partial v}(\eta, t) = 0 & \partial\Omega \times]0, T[\end{cases} \quad (12)$$

and $z_2(\mu, t)$ is the component state of the stable part of the system (1) given by

$$\begin{cases} \frac{\partial z_2}{\partial t}(\mu, t) = A_2 z_2(\mu, t) + (I - P)Bu(t) & \Omega \times]0, T[\\ z_2(\mu, 0) = z_{2_0}(\mu)\bar{\Omega} \\ \frac{\partial z_2}{\partial v}(\eta, t) = 0 & \partial\Omega \times]0, T[\end{cases} \quad (13)$$

The operator A_1 is represented by a matrix of order $(\sum_{i=1}^J m_i, \sum_{i=1}^J m_i)$ defined by $A_1 = \text{diag}[\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_j, \dots, \lambda_j]$ and $PB = [G_1^{tr}, G_2^{tr}, \dots, G_j^{tr}]$. By using the condition (2) of this theorem, we conclude that the suite $(D_i, f_i)_{1 \leq i \leq q}$ of sensors are Γ_G -strategic for the unstable part of the system (1), the sub-system (12) is an approximately Γ_G -observable [23]. Since this system it is of finite dimensional, then it is an exactly Γ_G -observable [22]. Therefore it is Γ_G -detectable [21], and hence there exists an operator $H_{\Gamma_G}^1$ such that $(A_1 - H_{\Gamma_G}^1 C)$ which satisfies the following [15, 23]:

$$\exists M_{\Gamma_G}^1, \omega_{\Gamma_G}^1 > 0, \text{ such that } \left\| e^{(A_1 - H_{\Gamma_G}^1 C)t} \right\|_{(H^{1/2}(\Gamma))^n} \leq M_{\Gamma_G}^1 e^{-\omega_{\Gamma_G}^1 t}$$

and we have

$$\|z_1(\cdot, t)\|_{(H^{1/2}(\Gamma))^n} \leq M_{\Gamma_G}^1 e^{-\omega_{\Gamma_G}^1 t} \|Pz_0(\cdot)\|_{(H^{1/2}(\Gamma))^n}.$$

Since the semi-group generated by the operator A_2 is Γ_G -stable on $(H^{1/2}(\Gamma))^n$, then there exist $M_{\Gamma_G}^2, \omega_{\Gamma_G}^2 > 0$, such that

$$\|z_2(\cdot, t)\|_{(H^{1/2}(\Gamma))^n} \leq M_{\Gamma_G}^2 e^{-\omega_{\Gamma_G}^2 t} \|(I - P)z_0(\cdot)\|_{(H^{1/2}(\Gamma))^n} + \int_0^t M_{\Gamma_G}^2 e^{-\omega_{\Gamma_G}^2 t} \|(I - P)z_0(\cdot)\|_{(H^{1/2}(\Gamma))^n} \|u(\tau)\| d\tau.$$

Therefore $z(\mu, t) \rightarrow 0$ when $t \rightarrow \infty$. Finally, the system (1) and (2) are Γ_G -detectable.

Conversely, if the systems (1)-(2) are Γ_G -detectable. Then there exists an operator $H_{\Gamma_G} \in L(R^q, (H^{1/2}(\Gamma))^n)$, such that $(A - H_{\Gamma_G} C)$ generates Γ_G -stable, strongly continuous semi-group $(S_{H_{\Gamma_G}}(t))_{t \geq 0}$ on the space $(H^{1/2}(\Gamma))^n$ which satisfies the following:

$$\exists M_{\Gamma_G}, \omega_{\Gamma_G} > 0, \text{ such that } \left\| \chi_{\Gamma} \gamma \nabla S_{H_{\Gamma_G}}(\cdot) \right\|_{(H^{1/2}(\Gamma))^n} \leq M_{\Gamma_G} e^{-\omega_{\Gamma_G} t}.$$

Thus, the unstable sub-system (12) is Γ_G -detectable. We recall that a system is an approximately Γ_G -observable, i.e.

$$[K\gamma^* \nabla \chi_{\Gamma}^* z^*(\cdot, t) = 0 \Rightarrow z^*(\cdot, t) = 0], \text{ for } z^*(\cdot, t) \in (H^{1/2}(\Gamma))^n,$$

we have

$$K\gamma^* \nabla \chi_{\Gamma}^* z^*(\cdot, t) = \left(\sum_{i=1}^J e^{\lambda_j t} \langle \psi_j(\cdot), \gamma^* \chi_{\Gamma}^* \nabla z^*(\cdot, t) \rangle_{(H^{1/2}(\Gamma))^n} \langle \psi_j(\cdot), f_j(\cdot, t) \rangle_{(H^{1/2}(\partial\Omega))^n} \right)_{1 \leq j \leq q}$$

If the rank $G_i z_i \neq m_i$ for all $i, i = 1, \dots, J$, there exists $z^*(\cdot, t) \in (H^{1/2}(\Gamma))^n$, such that $K\gamma^* \nabla \chi_{\Gamma}^* z^*(\cdot, t) = 0$, this leads

$$\sum_{j=1}^J \langle \psi_j(\cdot), z^*(\cdot, t) \rangle_{(H^{1/2}(\Gamma))^n} \langle \psi_j(\cdot), f_j(\cdot, t) \rangle_{(H^{1/2}(\Gamma))^n} = 0.$$

The state vectors z_i may be given by

$$z_i(\cdot, t) = [\langle \psi_j(\cdot), z^*(\cdot, t) \rangle_{(H^{1/2}(\Gamma))^n} \langle \psi_j(\cdot), z^*(\cdot, t) \rangle_{(H^{1/2}(\partial\Omega))^n}]^{tr} \neq 0$$

we obtain $G_i z_i = 0$ for all $i, i = 1, \dots, J$

Consequently, the sub-system (12) is not an approximately Γ_G -observable [23] and therefore the suite $(D_i, f_i)_{1 \leq i \leq q}$ of sensors is not Γ_G -strategic. Thus, the systems (1)-(2) are not Γ_G -detectable [22]. Finally, we have $\text{rank} G_i \neq m_i$ for all $i, i = 1, \dots, J$. \square

3. Internal and Regional Boundary Detection of State Gradient Approach

In this section, we show that it is possible to link the internal regional gradient detectability and regional boundary gradient detectability, if we consider the following

• Let $\mathcal{R}: (H^{1/2}(\Gamma))^n \rightarrow (H^1(\Omega))^n$, which is continuous and linear given by [28]

$$\chi_{\Gamma} \gamma \nabla \mathcal{R} h(\mu, t) = h(\mu, t), \quad \text{for all } h \in (H^{1/2}(\Gamma))^n \quad (14)$$

• Let $E = \bigcup_{z \in \Gamma} B(z, r)$ and $\bar{\omega}_r = E \cap \bar{\Omega}$, where $B(z, r)$ is the ball of radius r centered in $z(\mu, t)$ and Γ is a part of $\bar{\omega}_r$ ($r > 0$ is an arbitrary and sufficiently small real) see (Figure 3).

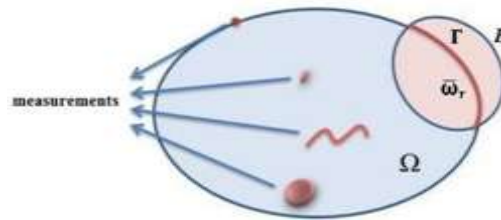


Fig. 3. The domain Ω , sub-domain $\bar{\omega}_r$, and the region Γ .

Now, we show that there exists a link between the Γ_G -strategic sensor and $\bar{\omega}_{rG}$ -strategic sensor in the following result.

Proposition 3.1: From the above results, we deduce that.

- 1- A sensor is Γ_G -strategic if the sensor is $\bar{\omega}_{rG}$ -strategic.
- 2- A sensor is Γ_G -strategic if the system is exactly $\bar{\omega}_{rG}$ -observable.

Proof 1: Since the sensor is $\bar{\omega}_{rG}$ -strategic in $\bar{\omega}_r$, this mean that the system is approximately $\bar{\omega}_{rG}$ -observable in $\bar{\omega}_r$.

Thus, the system is approximately Γ_G -observable [23]. Therefore, the sensor is Γ_G -strategic. \square

Proof 2: Let $z(\mu, t) \in (H^{1/2}(\Gamma))^n$ and $\bar{z}(\mu, t)$ be an extension to $(H^{1/2}(\partial\Omega))^n$. By using equation (11) and trace theorem there exists $\mathcal{R}\bar{z}(\mu, t) \in (H^1(\Omega))^n$, with bounded support such that

$$\gamma \mathcal{R} \bar{z}(\mu, t) = \bar{z}(\mu, t).$$

Since the system is an exactly $\bar{\omega}_{rG}$ -observable, then the system is approximately $\bar{\omega}_{rG}$ -observable [3]. And, since a system is approximately $\bar{\omega}_{rG}$ -observable then a system is approximately Γ_G -observable [1, 21-23] Thus, the sensor is Γ_G -strategic. \square

Corollary 3.2: From the previous results, we have.

- 1- If the system is an exactly $\bar{\omega}_{rG}$ -observable, then the system is exactly Γ_G -observable, i.e., there exists an operator $\chi_{\bar{\omega}_r} \nabla K^*: \mathcal{O} \rightarrow (H^1(\omega_r))^n$ given by

$$H_{\bar{\omega}_r} y(\cdot, t) = \chi_{\bar{\omega}_r} \nabla K^* y(\cdot, t) = \chi_{\bar{\omega}_r} \mathcal{R} \bar{z}(\mu, t).$$

Hence,

$$\chi_{\Gamma} \left(\gamma \chi_{\bar{\omega}_r} \nabla K^* y(\cdot, t) \right) = z(\mu, t).$$

where $z(\mu, t) \in (H^{1/2}(\Gamma))^n$ and $\bar{z}(\mu, t)$ be an extension to $(H^{1/2}(\partial\Omega))^n$.

- 2- If the system is approximately $\bar{\omega}_{rG}$ -observable, then the system is approximately Γ_G -observable.

Definition 3.3: The system (1) is ω_{rG} -stable, if the solution of autonomous system associated with (1) together with (2) converges exponentially to zero when $t \rightarrow \infty$.

Definition 3.4: The systems (1)-(2) are ω_{r_G} -detectability, if there exists an operator $H_{\omega_{r_G}}: \mathcal{O} \rightarrow (H^1(\omega_r))^n$, such that the operator $(A - H_{\omega_{r_G}} C)$ generates a strongly continuous semi-group $(S_{H_{\omega_{r_G}}}(t))_{t \geq 0}$, which is ω_{r_G} -stable.

Proposition 3.5: If the systems (1)-(2) are an exactly $\bar{\omega}_{r_G}$ -observable, then it is Γ_{EG} -detectable.

Proof: For the proof see ref. [15], with minor changment.

Now, the method of crossing from internal ω_{r_G} -detectability into Γ_G -detectability will be given in the following theorem.

Theorem 3.6: If the systems (1)-(2) are $\bar{\omega}_{r_G}$ -detectable, then it is Γ_G -detectable.

Proof: Let $z(\mu, t) \in (H^{1/2}(\Gamma))^n$ and $\bar{z}(\mu, t)$ be an extension to $(H^{1/2}(\partial\Omega))^n$ [28]. By using equation (14) and trace theorem, there exist

$$\mathcal{R}\bar{z}(\mu, t) \in (H^1(\Omega))^n$$

with a bounded support, such that

$$\gamma(\mathcal{R}\bar{z}(\mu, t)) = \bar{z}(\mu, t)$$

since the systems (1)-(2) are $\bar{\omega}_{r_G}$ -detectable, then it is ω_{r_G} -detectable. Thus, there exists an operator

$$\chi_{\omega_r} \nabla K^*: \mathcal{O} \rightarrow (H^1(\omega_r))^n$$

is defined by

$$H_{\omega_{r_G}} y(\cdot, t) = \chi_{\omega_r} \nabla K^* y(\mu, t)$$

such that the operator $(A - H_{\omega_{r_G}} C)$ generates a strongly continuous semi-group $(S_{H_{\omega_{r_G}}}(t))_{t \geq 0}$, which is ω_{r_G} -stable. For every $y \in \mathcal{O}$, then we obtain

$$\chi_{\omega_r} \nabla K^* y(\mu, t) = \chi_{\omega_r} \nabla \mathcal{R}\bar{z}(\mu, t)$$

and hence

$$\chi_\Gamma(\gamma \nabla \chi_{\omega_r} K^* y)(\cdot, t) = z(\mu, t)$$

Consequently, there exists an operator

$$H_{\Gamma_G} = \chi_\Gamma(\gamma \nabla \chi_{\omega_r} K^* y): \mathcal{O} \rightarrow (H^{1/2}(\Gamma))^n,$$

such that $(A - H_{\Gamma_G} C)$ generates a semi-group $(S_{H_{\Gamma_G}}(t))_{t \geq 0}$, which is Γ_G -stable.

Finally, the systems (1)-(2) are Γ_G -detectable [15].

4. Application to Sensors Locations

In this section, we will explore different results related to different types of measurements and, we give the results on the locations of internal and boundary (pointwise and zone). Consider the two dimensional of diffusion system on a rectangular domain will may be described by the following form

$$\begin{cases} \frac{\partial z}{\partial t}(\mu_1, \mu_2, t) = \frac{\partial^2 z}{\partial \mu_1^2}(\mu_1, \mu_2, t) + \frac{\partial^2 z}{\partial \mu_2^2}(\mu_1, \mu_2, t) + z(\mu_1, \mu_2, t) & \Omega \times]0, T[\\ z(\mu_1, \mu_2, 0) = z_0(\mu_1, \mu_2) \bar{\Omega} & \\ \frac{\partial z}{\partial \nu}(\eta_1, \eta_2, t) = 0 & \partial\Omega \times]0, T[\end{cases} \quad (15)$$

with measurements obtained by output function given as in (2), where $\Omega =]0, a_1[\times]0, a_2[$, $\Gamma =]0, a_1[\times \{a_2\}$ and the eigenfunctions of the dynamic system (15) for Dirichlet boundary conditions are given by

$$\varphi_{nm}(\mu_1, \mu_2) = \left(\frac{4}{a_1 a_2}\right)^{1/2} \cos n\pi \left(\frac{\mu_1}{a_1}\right) \cos m\pi \left(\frac{\mu_2}{a_2}\right) \quad (16)$$

associated with eigenvalues

$$\lambda_{nm} = -\left(\frac{n^2}{a_1^2} + \frac{m^2}{a_2^2}\right) \pi^2, \quad n, m \geq 1 \quad (17)$$

If we suppose that $a_1^2/a_2^2 \notin Q$, then the multiplicity of the eigenvalues λ_{nm} is $r_{nm} = 1$ for every $n, m = \{1, \dots, J\}$, then one sensor (D, f) may be sufficient for Γ_G -detectable [16].

4.1 Case of Zone Sensor

We study and discuss the following cases.

• Internal Zone Sensor

Consider the system (15)-(2) may be given by the form

$$y(t) = \int_D z(\mu_1, \mu_2, t) f(\mu_1, \mu_2) d\mu_1 d\mu_2 \quad (18)$$

where the zone sensor is the located inside the domain Ω , over the support

$$D =]\mu_{1_0} - l_1, \mu_{1_0} + l_1[\times]\mu_{2_0} - l_2, \mu_{2_0} + l_2[\subset \Omega \in L^2(D)$$

as in (figure 4). Now, from the above result, we give the following result.

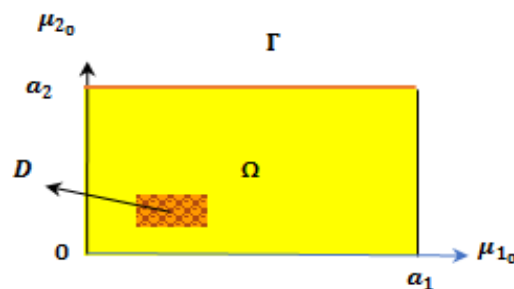


Fig. 4: Location D of

internal zone sensor.

Proposition 4.1: If the respect to $\mu_i = \mu_{i_0}, i = 1, 2$, then the systems (15)-(18) are Γ_G -detectable, if

$$n \mu_{1_0}/a_1 \text{ and } m \mu_{2_0}/a_2 \notin N, \text{ for all } n, m = \{1, \dots, J\}.$$

function f_i is symmetric with

• Boundary Zone Sensor

Now, the measurements are given by the output

$$y(t) = \int_{\Gamma_0} \frac{\partial z}{\partial v}(\eta_1, \eta_2, t) f(\eta_1, \eta_2) d\eta_1 d\eta_2, \quad (19)$$

with $\Gamma_0 \subset \partial\Omega$ is the boundary support of the sensor and $f \in L^2(\Gamma_0)$. In the case, where the support of the sensor (D, f) is one of side as in (figure 5), then we have the following proposition.

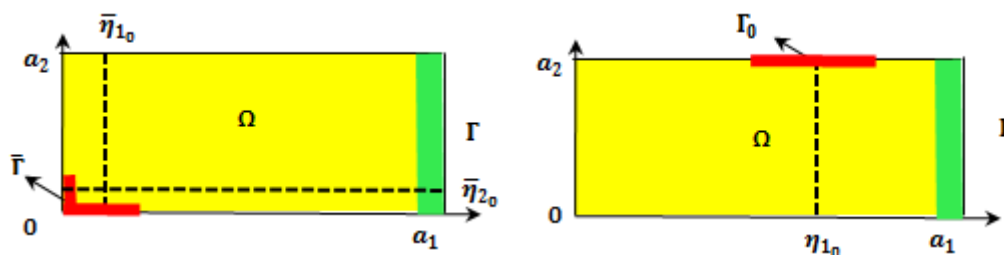


Fig. 5: Locations $\Gamma_0, \bar{\Gamma}$ of boundary zone sensors.

Proposition 4.2: If the function f is symmetric with respect to $\eta_1 = \eta_{1_0}$, then the systems (15)-(19) are Γ_G -detectable, if

$$n \eta_{1_0}/a_1 \notin N, \text{ for all } n, n = \{1, \dots, J\}$$

when the support of the sensor is on two sides, i.e., $\bar{\Gamma} = [0, \bar{\eta}_{1_0} + l_1] \times \{0\} \cup \{0\} \times [0, \bar{\eta}_{2_0} + l_2] = \Gamma_1 \cup \Gamma_2 \subset \partial\Omega$, as in (figure 5) we obtain the following result.

Proposition 4.3: If the function f_i is symmetric with respect to $\eta_i = \bar{\eta}_{i_0}, i = 1, 2$, then the systems (15)-(19) are Γ_G -detectable, if

$$2n \bar{\eta}_{1_0}/a_1 \text{ and } 2m \bar{\eta}_{2_0}/a_2 \notin N, \text{ for all } n, m = \{1, \dots, J\}.$$

4.2 Case of Pointwise Sensor

We investigate different pointwise cases.

• Internal Pointwise Sensor

In this case, we can give the output function by the following form

$$y(t) = \int_{\Omega} z(\mu_1, \mu_2, t) \delta(\mu_1 - b_1, \mu_2 - b_2) d\mu_1 d\mu_2 \quad (20)$$

where $b = (b_1, b_2)$ is the location of pointwise sensor in Ω as defined in (figure 6). Then we obtain the following result.

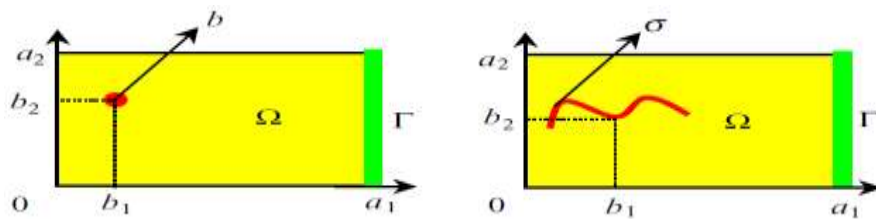


Fig. 6: Locations b, σ of internal pointwise and filament sensors.

Proposition 4.4: We conclude that.

1- Pointwise case: If the sensor is located in $b = (b_1, b_2)$, then the systems (15)-(20) are Γ_C -detectable, if

$$nb_1/a_1 \text{ and } mb_2/a_2 \notin N, \text{ for all } n, m = \{1, \dots, J\}.$$

2- Filament case: Suppose that the observation is given by the filament sensor $\sigma = \text{Im}(\gamma)$ is symmetric with respect to the line $b = (b_1, b_2)$, if

$$nb_1/a_1 \text{ and } mb_2/a_2 \notin N, \text{ for all } n, m = \{1, \dots, J\}.$$

• Boundary pointwise sensor

Suppose that the sensor (b, δ_b) is located on b , where $b = (b_1, b_2) \in \partial\Omega$ with $b = (0, b_2)$ as in (Figure 7).

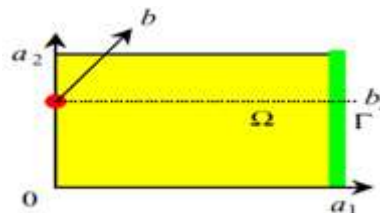


Fig. 7: Location b of boundary pointwise sensor.

The output function is given by

$$y(t) = \int_{\partial\Omega} \frac{\partial z}{\partial v}(0, \eta_2, t) \delta(0, \eta_2 - b_2) d\eta_1 \quad (21)$$

where $b = (0, b_2)$. Thus, we obtain the following result.

Proposition 4.5: If the sensor support $b \in \partial\Omega$, then systems (15)-(21) are not Γ_C -detectable, if

$$2nb_2/a_2 \notin N, \text{ for all } n, n = \{1, \dots, J\}.$$

Remark 4.6: We can extend previous results in the case of rectangular domain to the case of circular domain with the following system

$$\begin{cases} \frac{\partial z}{\partial t}(r, \theta, t) = \frac{\partial^2 z}{\partial r^2}(r, \theta, t) + \frac{\partial^2 z}{\partial \theta^2}(r, \theta, t) + z(r, \theta, t) & \Omega \times]0, T[\\ z(r, \theta, 0) = z_0(r, \theta) \bar{\Omega} & \\ \frac{\partial z}{\partial v}(a, \theta, t) = 0 & \partial\Omega \times]0, T[\end{cases} \quad (22)$$

augmented output function is defined by

$$y_n(t) = \int_{D_n} \frac{\partial z}{\partial v}(r_n, \theta_n, t) f(r_n, \theta_n) dr_n d\theta_n \quad (23)$$

where $2 \leq n \leq q$, $\Omega =]0, a[$, $r = a > 0$, $\theta \in [0, 2\pi]$, for zone and pointwise sensor may be internal or boundary.

5. Γ_G -Detectability and Γ_G -Observer

In this section, we show that the regional boundary gradient detectability is the possibility to defined a regional boundary gradient observer which is enable to estimate the state gradient of considered system in part Γ of boundary $\partial\Omega$. This approach is derived from the previous researchs as in [7, 13-16, 21].

Definition 5.1: Suppose that there exists dynamical system with state $w(\mu, t) \in W$ given by

$$\begin{cases} \frac{\partial w}{\partial t}(\mu, t) = F_{\Gamma_G} w(\mu, t) + G_{\Gamma_G} u(t) + H_{\Gamma_G} y(\cdot, t) & \Omega \times]0, T[\\ w(\mu, 0) = w_0(\mu) \bar{\Omega} & \\ \frac{\partial w}{\partial \nu}(\eta, t) = 0 & \partial\Omega \times]0, T[\end{cases} \quad (24)$$

where F_{Γ_G} generates a strongly continuous semi-group, which is Γ_G -stable on the space W , $G_{\Gamma_G} \in L(U, W)$ and $H_{\Gamma_G} \in L(\mathcal{O}, W)$. The system (24) defines an Γ_G -estimator for $T_{\Gamma_G} z(\mu, t) = \chi_{\Gamma} \nabla T z(\mu, t)$, if the following conditions hold:

1- $\lim_{t \rightarrow \infty} [T_{\Gamma_G} z(\mu, t) - w(\mu, t)] = 0$, $\mu \in \Gamma$.

3- T_{Γ_G} maps $D(A)$ into $D(F_{\Gamma_G})$, where $z(\mu, t)$ and $w(\mu, t)$ are the solutions of systems (1)-(2) and (24).

Definition 5.2: The system (24) specifies an Γ_G -observer for the system (1) together with the output function (2), if the following conditions hold:

1- There exists $R_{\Gamma_G} \in L(\mathcal{O}, (H^{1/2}(\Gamma))^n)$ and $S_{\Gamma_G} \in L((H^{1/2}(\Gamma))^n)$, such that

$$R_{\Gamma_G} C + S_{\Gamma_G} T_{\Gamma_G} = I_{\Gamma_G}.$$

2- $T_{\Gamma_G} A - F_{\Gamma_G} T_{\Gamma_G} = H_{\Gamma_G} C$ and $G_{\Gamma_G} = T_{\Gamma_G} B$.

3- The system (14) defines an Γ_G -estimator for $T_{\Gamma_G} z(\mu, t)$.

Proposition 5.3: If the systems (1)-(2) are Γ_G -detectable, then the dynamical system (24) is Γ_G -observer of the systems (1)-(2), if

$$\lim_{t \rightarrow \infty} [z(\mu, t) - w(\mu, t)] = 0, \quad \mu \in \Gamma$$

Proof: Let $\varphi(\mu, t) = z(\mu, t) - w(\mu, t)$, where $w(\mu, t)$ is the solution of the system (24). Deriving the above equation and using the equations (1) and (24), we obtain

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(\mu, t) &= \frac{\partial z}{\partial t}(\mu, t) - \frac{\partial w}{\partial t}(\mu, t) \\ &= (A - H_{\Gamma_G} C)\varphi(\mu, t) \end{aligned}$$

The system (1) is Γ_G -detectable, there exists an operator $H_{\Gamma_G} \in L(\mathcal{O}, (H^{1/2}(\Gamma))^n)$, such that $(A - H_{\Gamma_G} C)$ generates a strongly continuous semi-group $(S_{H_{\Gamma_G}}(t))_{t>0}$, which is Γ_G -stable on $(H^{1/2}(\Gamma))^n$, there exists $M_{\Gamma_G}, \omega_{\Gamma_G} > 0$, such that

$$\|\varphi\|_{(H^{1/2}(\Gamma))^n} \leq \left\| \chi_{\Gamma} \gamma \nabla S_{H_{\Gamma_G}}(t) \right\|_{(H^{1/2}(\Gamma))^n} \|\varphi_0\| \leq M_{\Gamma_G} e^{-\omega_{\Gamma_G} t} \|\varphi_0\| \quad (25)$$

With

$$\varphi_0(\mu) = z_0(\mu) - w_0(\mu),$$

and hence, we have the following result

$$\lim_{t \rightarrow \infty} [z(\mu, t) - w(\mu, t)] = 0, \quad \mu \in \Gamma.$$

Definition 5.4: The systems (1)-(2) are Γ_G -observable exponentially, if there exists a dynamical system which is Γ_G -observer exponentially estimate the state gradient of the original system .

Definition 5.5: The system (24) is said to be an identity or full order Γ_G -observer for the systems (1)-(2) [4], if

$$T_{\Gamma_G} = I_{\Gamma_G} \quad \text{and} \quad Z = W.$$

Definition 5.6: The system (24) is said to be a reduced-order Γ_G -observer for the systems (1)-(2), if

$$Z = \mathcal{O} \oplus W \quad [29].$$

Remark 5.7: If the system is a regionally boundary gradient detectable, then it is possible to construct an regionally boundary gradient observer for the original system.

Acknowledgements. Our thanks in advance to the editors and experts for considering this paper to publish in this esteemed journal. The authors appreciate your time and effort in reviewing the manuscript and greatly value your assistant as reviewer for the paper.

6. Conclusion

We have explored the original results devoted to the concept of regional boundary gradient detectability to the state gradient for parabolic distributed system in Hilbert space. Then, we have shown that, the possibility to design a dynamic system which is enable to estimate the state gradient in sub-region Γ of the boundary $\partial\Omega$ by using detectability and strategic sensors in different situations. Moreover, many problem still open like the development of these results to case of hyperbolic distributed parameter systems as in [25].

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